The effects of public transportation on urban form

Leonardo J. Basso
Departamento de Ingeniería Civil, Universidad de Chile ljbasso@ing.uchile.cl

Matías Navarro
Centro de Desarrollo Urbano Sustentable (CEDEUS), minavar1@uc.cl

Hugo E. Silva
Departamento de Ingeniería de Transporte y Logística – Instituto de Economía, Pontificia Universidad Católica de Chile husilva@uc.cl

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Abstract

This paper proposes a spatial equilibrium model, in a monocentric city, which studies the interaction between the public transportation system, the housing market and the urban structure. Unlike most of the literature, we analyze the urban form of cities where people only ride on public transportation. This formalizes an urban form in the monocentric framework, where population density, price of housing, land rent, size of dwellings and the height of buildings do not follow a monotonic pattern as opposed to what has been found in previous literature. Our results suggest that, when the available mode of transportation is public transit, the price of housing and land rent are more expensive near stations and cheaper between them. The model also shows that the size of dwellings will be smaller near stations, where buildings will be taller than in other locations along the city. We analyze how the urban form changes with some key aspects of public transportation. We find that a higher fare or public transport stops that are more distant to each other has to be compensated with lower prices, smaller buildings and a closer city boundary to the CBD. On the other hand, shorter boarding times at each station due to an increase in public transport frequency is compensated with higher prices, taller buildings and a more distant city boundary. Finally, our model predicts a location pattern for income groups of imperfect segmentation. We show that public transportation provides the possibility of having mixed neighborhoods. That is, a city where poor and rich neighborhoods mix and intertwine, a pattern that is observed in reality and for which we provide an economic rationale.

1. Introduction

Transportation and especially commuting has been at the core of the study of urban land use. The seminal works of Alonso (1964), Mills (1967) and Muth (1969), shaped the land use analysis using the so-called monocentric city model. The model represents land as a line that has the central business district in its origin and endogenously determines population density, consumption and price of housing, rental price of land, and size of dwellings along the city. The core of the model is that people besides consuming a numeraire good, consume housing and must commute. As commuting costs increase with distance to the CBD, the differences of these costs along the city must be balanced by differences in the price of living space (Brueckner, 1987). As a consequence, the price of housing must decrease with distance to the CBD. The model also predicts decreasing gradients of prices of land and structural density (an index of building heights) and an increasing gradient of dwelling sizes. [1]

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[1] See Brueckner (1987) for an exposition of the monocentric city model and main implications.

A second aspect of urban economics that is usually studied with this model is the pattern of residential sorting. When extending the analysis to heterogeneity in income, e.g. by considering two income groups, the model is also able to determine how income groups sort themselves in the city. The main prediction of the model is that there is perfect segmentation of income groups. In the simplest case of two income groups, rich and poor, the group with the steepest bid-rent curve lives close to the CBD, as it has a higher willingness to pay, and the other locates in the suburbs. Which of the two groups is located close to the CBD and which one in the suburbs depends on the relative magnitude of the income elasticity of commuting costs (per unit of distance) and the income elasticity of demand for housing. While Wheaton (1973) finds that both elasticities are roughly equal, Glaeser, Kahn and Rappaport (2008) argue that the elasticity of commuting costs is 1 and the elasticity of demand for land is less than one, a result that implies in the monocentric model that the rich should live in the urban core and the poor in the suburbs. However, the assumption that the elasticity of commuting costs is 1 is arguably too strong and the empirical results have been questioned as well (see, e.g., Puga and Duranton, 2014).[2]

In this paper we seek to understand what the effect of public transportation on urban form is. We study the effect on the gradients of prices, structural density and dwelling size, and the effect on residential sorting of (two) income groups. We also study the effect of different aspects of public transportation such as pricing schemes on city size, population density and all the other variables described above. By formalizing the modelling of public transportation commuting and assuming that is the only available motorized mode, we show that all the gradients are non-monotonic and have peaks at public transport stations. In particular, the price of housing and the height of the buildings have a peak at stations and these peaks are decreasing with distance. The size of dwellings have a peak in between stations and these peaks are growing with distance to the CBD. The main mechanism behind these results is the fact that people have to walk to the station in order to take public transportation. Therefore, commuting costs do not increase monotonically with distance (as people sometimes walk to stations that are further away from the CBD). While this non-monotonicity due to walking to stations has been mentioned before (see Duranton and Puga, 2014, p.16), to the best of our knowledge, this is the first paper to formalize it.

The main contribution of the paper is to provide a new explanation for partial residential sorting and to assess the effect of public transportation variables (such as pricing schemes) on the urban form. We show that regardless of whether the elasticity of commuting costs is greater or less than the elasticity of demand for land, imperfect segmentation of income groups can always occur. That is, that the presence of public transit alone provides a rationale for mixed neighborhoods. For example, if the income elasticity of commuting costs is smaller than the income elasticity of the demand for land, there is a zone next to the CBD where poor people locate and a zone close to the edge of the city, the suburbs, where the rich live, just as in the standard model. The difference is that in between these two zones there may be several zones in which the poor and the rich alternate locations. We show that this mixing can happen between the CBD and the first public transport station, so that what could be called the inner city has people of both income types while the suburbs have only rich people. The mixing in the inner city also happens when the income elasticity of commuting costs is larger than the income elasticity of the demand for land, but the poor are located in the suburbs (and the rich next to the CBD). These two patterns, observed in many cities, can be explained by the sole presence of public transportation.

The effects of the key variables of public transport on the urban form are as follows. We find that a higher fare or stops that are more distant to each other has to be compensated with lower prices, smaller buildings and a city boundary closer the CBD. On the other hand, shorter boarding times at

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[2] See Duranton and Puga (2014) for a treatment of the monocentric model, its implications for residential sorting and a discussion of empirical evidence.

each stations as a result of increased public transportation frequency is compensated with higher prices, taller buildings and a more distant city boundary.

We are certainly not the first to extend the monocentric model. Extensions have focused on explaining the size of urban areas (Mills & de Ferranti, 1971; Sheshinski, 1973; Henderson, 1975), land rents behavior along the city (Solow, 1972; Wheaton, 1974), land allocation for housing and roads (Solow & Vickrey, 1970; Solow 1973), and the effects of congestion on the urban form (Kanemoto, 1980; Arnott, 1979). Subsequent works in this line of research (Sullivan, 1983; Brueckner, 1995; Verhoef, 2005) have dedicated their efforts to study the benefits of using second-best road pricing and other policies. More recently, Gubins & Verhoef (2014) have incorporated a dynamic model of congestion to the analysis – Vickrey's (1969) bottleneck model– in order to examine the interaction between trip timing decisions and household location.

Despite that these studies have given valuable insights to the urban economics literature, their results have a strong limitation: almost all of them consider only one available mode of transportation: private cars. The main article which mentions public transport is LeRoy and Sonstelie (1983). Based on the Alonso's model, households choose between transport modes. One of them is cheaper –in fixed and variable costs– but slower, and the other one is more expensive but faster. Also, there are two income groups who, due to their different willingness to pay for innovations in transport technologies, choose different locations to live. Although they provide an interesting explanation for location patterns of income groups, they do not effectively model public transport and are not able to analyze the impact of different pricing schemes or any other aspect of public transportation on urban form. Our hypothesis is that including public transportation and its interaction with the housing market may add new insights to current literature conclusions about urban form.

The paper is organized as follows. The section 2 presents the basic model of a city where homogeneous consumers exclusively commute by public transportation. The section 3 extends the previous section, complementing the basic model for the case of heterogeneous consumers. The section 4 discusses the main conclusions of this work.

2. The basic model for homogeneous consumers

We base our analysis in a monocentric city, in which each resident commutes to a job in the CBD along a radial network of public transportation. Commuting costs are the sum of the public transport trip fare, e, and travel time costs. We assume that consumer are homogeneous in preferences and income; in Section 3 we extend the analysis and allow for heterogeneity. Note that the basic model, assumptions and exposition follow Brueckner (1987).

2.1. Consumer side and first-order conditions

Residents earn y per period and their preferences are represented by a common strictly quasi-concave utility function $U(c, q)$, where c is consumption of a composite non-housing good and q is consumption of housing (measured in square meters of floor space). Following Brueckner (1987) we assume that the price of the composite good is the same everywhere in the city (normalized to unity for simplicity) and that the rental price per square meters of housing floor space, p, varies with the distance to the CBD (x). The income y is spent on paying the consumption of non-housing goods c, on housing rent (pq), and on commuting. The generalized cost of commuting on public transportation, ρ, includes both the monetary costs (i.e. the fare of public transportation) and the time costs (i.e. the commuting time from x to the CBD). Due to the assumption that residents are identical, the urban equilibrium must yield identical utility level u for all individuals. Substituting for c in the utility function using the budget constraint $y = c + pq + \rho$, the requirement that the maximized utility level equals u can be written as,

$$\max_{\{q\}} V(y - pq - \rho, q) = u \tag{1}$$

Consumers face a tradeoff between housing and goods consumption. The consumer's choices can be characterized from the first order condition from equation (1). Since residents choose q optimally conditional on prices, the first-order condition is

$$\frac{V_q(y - pq - \rho, q)}{V_c(y - pq - \rho, q)} = p \tag{2}$$

Where V_m represents the partial derivative of the indirect utility function with respect to m, being m equals c or q. This is, the marginal rate of substitution between housing and non-housing goods must equal their price ratio.

Also, since residents must reach the same utility u in each location x, the spatial equilibrium condition which guarantees this requirement is

$$V(y - pq - \rho, q) = u \tag{3}$$

These two conditions allow us to obtain solutions for p and q, conditional on the parameters of the parameters x, y, ρ, and u.

2.2. Housing supply side and first-order conditions

Following Brueckner (1987), housing is produced with inputs of land L and capital N, according to the concave constant returns production function $H(N, L)$. This function gives the number of square meters of floor space contained in a building.

While the floor space is rented by residents at price p, each square meter of land and each unit of capital are rented by producers at prices r and i, respectively. Therefore, the producer's profit is $pH(N, L) - iN - rL$. Using the constant returns property of H we can rewrite the profit as $L \cdot (pH(N/L, 1) - iN/L - r)$. Denoting S the capital-land ratio N/L, the profit can be written as

$$L \cdot (ph(S) - iS - r) \tag{4}$$

where $h(S) \equiv H(S, 1)$ gives floor space per square meters of land. Note that S is an index of the height of buildings. Also, the marginal productivity of capital is positive $h'(S) > 0$ and $h''(S) < 0$ because, for a fixed L, the use of additional capital will be less productive (e.g. thicker walls, deeper foundations). The producers maximize profit per square meters of land for fixed L, choosing S. Using equation (4), the first-order condition is

$$ph'(S) = i \tag{5}$$

and the zero-profit condition is

$$ph(S) - iS = r \tag{6}$$

2.3. Public transportation modelling

Up to this point the model and assumptions are conventional. Our distinctive feature is the modelling of public transportation, which is reflected in the commuting costs. Previous studies have assumed

that commuting time costs are due to in-vehicle travel time and that it is an increasing function of the distance. While this is realistic for car travel, it is not for public transportation. Besides the in-vehicle travel time, commuters must walk to a station and wait for the vehicle.

We assume there are n desired arrival times at work which is equivalent to assume that there are n uniform groups of residents along the city, where each group has a different work arrival time. Each group of residents must reach the CBD at their arrival time at work commute on f public transportation vehicles. We consider a radial network of public transportation which has equally spaced stations at distance s. Public transportation is uncongested (e.g. train or subway) both on roads and at stations (i.e. there is not overcrowding), and the free-flow speed in public transportation is denoted v_f. Residents do not experiment waiting time, because public transportation operates based on time tables. Therefore, the only difference with previous analyses (Alonso, 1964; Dixit, 1973; Brueckner, 1987) is the walking costs.

Figure 1: Public transportation stations

Residents face the choice between walking to the CBD or to the nearest station and commute by public transport. To avoid the trivial solution in which everyone walks to the CBD we assume that parameters are such that the indifferent resident between walking and commuting by public transport is located between the CBD and the first station[3]. When residents live between the CBD and the first station, they have to decide if they directly walk downstream to the CBD or if they walk upstream to the station and commute by public transportation. Denote $\bar{x}^0$ the location for which residents are indifferent between both options because the generalized cost of commuting from the station located at s equals the generalized cost of walking to the CBD (see Figure 1). $\bar{x}^0$ can be obtained from equalizing commuting costs by both modes:

$$e^s + \theta \left[\frac{1}{v_f} \int_0^s ds + \frac{t_b}{nf} \int_{\bar{x}^0}^{\bar{x}^1} xD\,dx + \frac{s - \bar{x}^0}{v_c} \right] = \theta \frac{\bar{x}^0}{v_c} \tag{7}$$

The right side of equation (7) is the generalized cost of walking from $\bar{x}^0$ to the CBD at speed v_c, where θ is the value of time[4]. The left side of equation (7) represents the generalized cost of commuting from station at s. The first term is the public transportation fare charged at station s. The second term is the in-vehicle travel time between the first station and the CBD. The third term is the number of residents who board the public transportation at the first station. The parameter t_b is the boarding time per passenger which must be small enough to $\bar{x}^0$ be lower than s. The population density D is defined as square meters of floor space per square meters of land divided by square meters of floor space per dwelling, that is, $h(S)/q$. This expression indicates dwellings per square meters or – assuming that each one contains one person– persons per square meters. Finally, the fourth term is the walking time from $\bar{x}^0$ to the first station at s. From equation (7), we can get:

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[3] People would always choose walking if either the public transport fare is prohibitively high or if the walking speed is equal to or higher than public transport speed.

[4] Formally, θ is the willingness to pay to reduce travel time by a unit of time. As we do not model consumer's time allocation it is an exogenous parameter that converts time into money. In Section 2.6 we determine this value endogenously by including consumer's time consumption.

$$\bar{x}^0 = \frac{v_c}{2\theta}\left(e^s + \frac{\theta s}{v_f} + \frac{\theta t_b}{nf}\int_{\bar{x}^0}^{\bar{x}^1} xDdx\right) + \frac{s}{2} \tag{8}$$

Note that $\bar{x}^0$ is always upstream (to the right) of half of s and the necessary condition on parameters that we need to impose to be interior (downstream of the station) is: $\dfrac{v_c\left(e^s + (\theta t_b/nf)\int_{\bar{x}^0}^{\bar{x}^1} xDdx\right)}{\theta(1 - v_c/v_f)} < s$ where $0 < \dfrac{v_c}{v_f} < 1$.

When residents live between two stations, they consider the differences in fare, in-vehicle travel time and walking time when choosing the boarding station. The left-hand side of the equation (9) is the generalized cost of commuting from the station k located at ks and the right-hand side represents the generalized cost of commuting from the station located at $(k+1)s$ where $k = 1, 2, 3$, and so on.

$$e^{ks} + \theta\left[\frac{1}{v_f}\int_0^{ks} ds + \frac{t_b}{nf}\int_{\bar{x}^0}^{\bar{x}^k} xDdx + \frac{\bar{x}^k - ks}{v_c}\right] = e^{(k+1)s} + \theta\left[\frac{1}{v_f}\int_0^{(k+1)s} ds + \frac{t_b}{nf}\int_{\bar{x}^0}^{\bar{x}^{k+1}} xDdx + \frac{(k+1)s - \bar{x}^k}{v_c}\right] \tag{9}$$

The solution for $\bar{x}^k$ can be obtained recursively according to the equation (10). As the value of $\bar{x}^0$ is totally determined by known parameters, the following indifference location (i.e. in this case, $\bar{x}^1$) can be obtained. This procedure can be repeated until the edge of the city.

$$\bar{x}^k = \frac{v_c}{2\theta}\cdot\left([e^{(k+1)s} - e^{ks}] + \theta\left[\frac{s}{v_f} + \frac{t_b}{nf}\int_{\bar{x}^k}^{\bar{x}^{k+1}} xDdx\right]\right) + \frac{s}{2}(2k+1) \tag{10}$$

On one hand, walking time depends on where people live, reaching its maximum value at $\bar{x}^k$, and decreasing nearer the CBD or the stations. On the other hand, the travel time on vehicle is updated (upward) at each neighborhood –delimited by the indifference locations $\bar{x}^k$– of each station along the city. Adding both terms, we obtain the total commuting time on public transportation t which increases from downstream stations until $\bar{x}^k$ and decreases from $\bar{x}^k$ until the next station (see Figure 2).
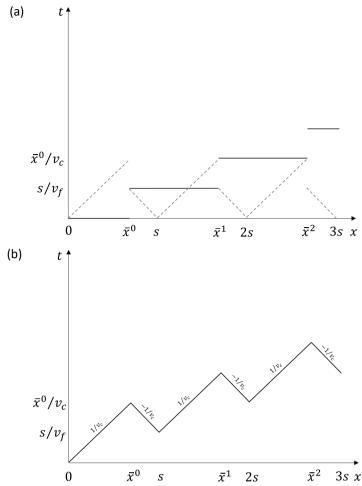
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Figure 2: Graphic (a) Time in-vehicle (solid lines) and walking time (dashed lines) and Graphic (b) Total commuting time.

3. Effects on urban form of cities

Public transportation and its modelling may have impact in urban variables such as p, q, r and S. In this section we present two analyses. On one hand, we show how these urban variables change along the city. On the other hand, we display the effect on urban form of public transportation variables such as spacing of stations, fare and frequency.

3.1. Intracity analysis

To describe the behavior of prices, dwelling sizes and buildings height we totally differentiated the conditions obtained for the consumer side (equation (2) and (3)) and for the supply side (equations (5) and (6)) with respect to x, ρ, y, and u. From these equations, we can describe how the prices p, q, r, and the structural density S changes along the city. While the results obtained for derivatives with respect to ρ, y, and u are analogous (the same sign) as those in Brueckner (1987), the outcomes from the variables along the city are novel.

Specifically, differentiating equation (3) with respect to x, yields

$$V_c \left(-q \frac{\partial p}{\partial x} - p \frac{\partial q}{\partial x} - \frac{\partial \rho}{\partial x} \right) + V_q \frac{\partial q}{\partial x} = 0 \tag{11}$$

Using that $V_q = pV_c$ (see equation (2)), we can solve for $\frac{\partial p}{\partial x}$. To keep the analysis simple we assume that the fare of public transportation is constant along the city or, at least, between indifference locations. This implies that the partial derivative of e with respect to x equals zero.

$$\frac{\partial p}{\partial x} = -\frac{1}{q} \frac{\partial \rho}{\partial x} = \begin{cases} ks < x < \bar{x}^k \to \frac{\partial \rho}{\partial x} > 0 \to \frac{\partial p}{\partial x} < 0 \\ \bar{x}^k < x < (k+1)s \to \frac{\partial \rho}{\partial x} < 0 \to \frac{\partial p}{\partial x} > 0 \end{cases} \tag{12}$$

While the model with only private cars predicts that the rental price per square meters of housing floor space constantly decreases from the CBD to the edge of the city, we obtain a different result. The difference lies in that the commuting cost does not increase monotonically with distance because walking time does not. The partial derivative of the generalized cost of commuting on public transportation ($\partial \rho / \partial x$) is negative or positive depending on whether the resident's location is upstream or downstream of the station. This implies that the partial derivative of p with respect to x changes its sign, being negative between any $\bar{x}^k$ and its upstream station, and positive between any station and the upstream $\bar{x}^k$. Therefore, the rental price per square meters of housing is higher near stations and lower between them (see Figure 3).

In the model with only private cars, residents faced a trade-off between rental price of housing and generalized cost of commuting, which permanently increases from the CBD to the city boundary. However, the generalized cost of commuting does not always has a positive slope along the city in our model with public transport. In this case, commuting times depend on resident's location, because residents must walk–more or less–to reach the stations of public transportation.

Differentiating equation (2) with respect to x yields

$$\frac{\partial \left(\frac{V_q}{V_c} \right)}{\partial q} \frac{\partial q}{\partial x} = \frac{\partial p}{\partial x} \tag{13}$$

Combining equations (12) and (13) yields

$$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial x} \left[\frac{\partial \left(\frac{V_q}{V_c} \right)}{\partial q} \right]^{-1} = \frac{\partial p}{\partial x} \eta = \begin{cases} ks < x < \bar{x}^k \to \frac{\partial p}{\partial x} < 0 \to \frac{\partial q}{\partial x} > 0 \\ \bar{x}^k < x < (k+1)s \to \frac{\partial p}{\partial x} > 0 \to \frac{\partial q}{\partial x} < 0 \end{cases} \tag{14}$$

The partial derivative of q with respect to x depends on η and $\partial p / \partial x$. On one hand, we already showed that the sign of $\partial p / \partial x$ can be positive or negative, depending on the location of x. On the other hand, η is negative, because the marginal rate of substitution between housing and goods decreases as the consumption of housing increases. Therefore, the size of dwellings will be larger near $\bar{x}^k$ and smaller near stations (see Figure 3).
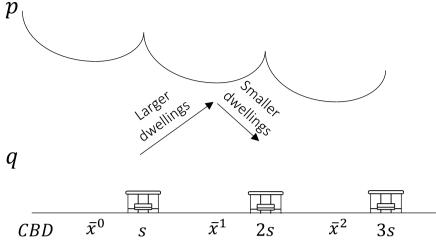
Figure 3: Variables p and q along the city

The convexity of p comes from its second partial derivative with respect to x ($\partial^2 p/\partial x^2$) which is positive[5] due to the identical sign between $\partial q/\partial x$ and $\partial \rho/\partial x$ at each location x.

Differentiating equation (6), with respect to x, and combining with (5) yields

$$\frac{\partial r}{\partial x} = h(S)\frac{\partial p}{\partial x} = \begin{cases} ks < x < \bar{x}^k \to \dfrac{\partial p}{\partial x} < 0 \to \dfrac{\partial r}{\partial x} < 0 \\[2mm] \bar{x}^k < x < (k+1)s \to \dfrac{\partial p}{\partial x} > 0 \to \dfrac{\partial r}{\partial x} > 0 \end{cases} \tag{15}$$

Remember that $h(S)$ gives the floor space per square meters of land and is always positive. Therefore, the sign of the partial derivative of r with respect to x has the same sign as the sign of the partial derivative of p with respect to x. That is, rent per square meters of land r will be higher near stations and lower between them (see Figure 4).

Finally, differentiating equation (5), with respect to x, yields

$$\frac{\partial S}{\partial x} = -\frac{\frac{\partial p}{\partial x}h'(S)}{ph''(S)} = \begin{cases} ks < x < \bar{x}^k \to \dfrac{\partial p}{\partial x} < 0 \to \dfrac{\partial S}{\partial x} < 0 \\[2mm] \bar{x}^k < x < (k+1)s \to \dfrac{\partial p}{\partial x} > 0 \to \dfrac{\partial S}{\partial x} > 0 \end{cases} \tag{16}$$

We showed above that marginal productivity of capital ($h'(S)$) is positive and its second derivative ($h''(S)$) is negative. The ratio between both expressions gives a negative sign which cancels the negative sign at the beginning of the right side of the equation (18). Therefore, the sign of the partial derivative of S with respect to x mainly depends on the sign of the partial derivative of p with respect to x. That is, buildings will be higher near stations and smaller between them (see Figure 4). This phenomena can be qualitatively observed in cities where subway is intensively used.

The convexity of r comes from its second partial derivative with respect to x ($\partial^2 r/\partial x^2$) which is positive[6] due to the identical sign between $\partial p/\partial x$ and $\partial S/\partial x$ at each location x along the city.

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[5] $\dfrac{\partial^2 p}{\partial x^2} = \dfrac{1}{q^2}\dfrac{\partial q}{\partial x}\dfrac{\partial \rho}{\partial x} - \dfrac{1}{q}\dfrac{\partial^2 \rho}{\partial x^2} > 0$

[6] $\dfrac{\partial^2 r}{\partial x^2} = \dfrac{\partial^2 p}{\partial x^2}h(S) + \dfrac{\partial p}{\partial x}h'(S)\dfrac{\partial S}{\partial x} > 0$
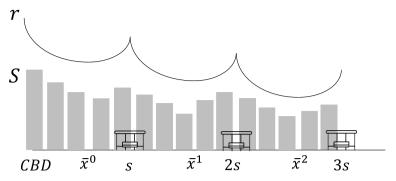
Figure 4: Variables r and S along the city

3.2. Comparative static analysis for public transportation variables

Public transportation has features such as spacing between stations s, fare e and number of vehicles f which may yield new effects on the urban form. These features open a new line of research about public transportation policies, which may be relevant for authorities or city planners who are responsible in their designing or managing. Particularly, our model allows us to investigate their effect on urban form variables such as the city boundary $\bar{X}$ and the height of buildings S.

This analysis requires to add two conditions that characterize the overall equilibrium of the urban area. The first requirement is that urban land rent r equals the agricultural rent r_A at the city boundary $\bar{X}$. The second requirement is that urban population L fit inside the city. Both conditions are expressed in the equations (17) and (18), respectively.

$$r(\bar{X}, y, \rho, u) = r_A \tag{17}$$

$$\int_0^{\bar{X}} xDdx = L \tag{18}$$

Our analysis assumes the open-city case where the exogenous parameters are u, r_A and y. In this case, we hold u fixed and we seek to understand the impact of changes in the public transportation features s, e and f on $\bar{X}$ and S. Since u is a fixed parameter, the only one variable affected by changes in s, e and f is the generalized cost of commuting ρ. Therefore, the impact on $\bar{X}$ and S follows immediately from equation (17).

Differentiating the equation (5) and (6) –the housing supplier conditions– with respect to $\varphi = \{e, s, f\}$, we can obtain the partial derivative of r and S with respect to these features. These partial derivatives are shown in the equations (19) and (20).

$$\frac{\partial r}{\partial \varphi} = h(S)\frac{\partial p}{\partial \varphi} \tag{19}$$

$$\frac{\partial S}{\partial \varphi} = -\frac{\frac{\partial p}{\partial \varphi}h'(S)}{ph''(S)} \tag{20}$$

As we can observe, both equations depend on the partial derivative of the rental price of housing with respect to φ. The expression which describes these relations is obtained from differentiating the equation (3) –the spatial equilibrium condition– with respect to φ. These partial derivatives are shown in the equation (21).

$$\frac{\partial p}{\partial \varphi} = -\frac{1}{q}\frac{\partial \rho}{\partial \varphi} \qquad (21)$$

Regarding the fare of public transportation, as the partial derivative of the generalized cost of commuting with respect to the fare is one[7], it follows from the equation (20) that the rental price of housing decreases with e. Therefore, from equations (19) and (20) we deduce that r and S decrease with e. That is, the city boundary and the height of buildings decrease when the fare of public transportation increases.

The sign of partial derivative of the generalized cost of commuting with respect to the spacing of stations can be analyzed intuitively. First, between the CBD and the first station, it is straightforward to show that $\bar{x}^0$ moves to the right (further away from the CBD). This happens because the cost of walking from the previous $\bar{x}^0$ to the CBD is the same as before, but now riding on public transportation implies more walking time and commuting in-vehicle time. Then, for residents who live between the CBD and $\bar{x}^0$ nothing changes, but for those who live from $\bar{x}^0$ and henceforward the generalized cost of commuting should increase. For the rest of the city, the generalized cost of commuting is composed by the fare, the time in-vehicle, the delay time at stations (due to residents' boarding) and the walking time to stations. An increasing in the spacing between stations implies an increase in the last two mentioned time-terms. While more spacing between stations yields a balance of less stations but more accumulation of residents for boarding at each one, both the average commuting time in-vehicle and the walking distance increase. Therefore, the partial derivative of the generalized cost of commuting with respect to spacing is positive which implies the rental price of housing will decrease when spacing between stations increases.

As the partial derivative of the rental price of housing with respect to the spacing between stations is negative, we now can conclude that both the land rent and the height of buildings will decrease when the spacing increases. That is, the city boundary will decrease and the city is smaller than before.

Finally, regarding to the number of public transportation vehicles, it is easy to realize that the generalized cost of commuting will decrease when this parameter increases. This happens because the boarding time at each station decreases due to fewer residents will board each vehicle. Hence, the sign of the partial derivative of the rental price of housing with respect to f is positive. That is, the city boundary and the height of buildings will increase when the number of public transportation vehicles is higher.

The results obtained are intuitive. On one hand, a higher commuting cost through a higher fare or stops that are more distant to each other has to be compensated with lower prices, smaller buildings and a closer city boundary to the CBD. On the other hand, a lower commuting cost due to shorter

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[7] $\frac{\partial \rho}{\partial e} = \frac{\partial e}{\partial e} + \theta \frac{\partial t}{\partial e} = 1$

boarding times at each stations is compensated with higher prices, taller buildings and a more distant city boundary.

4. Extensions to the basic model

In this section we present two extensions for the basic model. First, we show the results obtained when we add a time constraint in the consumer maximization problem, comparing these results with those presented in the previous section. Second, we include heterogeneous consumers into the model and then analyze their location pattern along the city.

4.1. Considering the time constraint

Based on Alonso's model with only private cars, our basic model considers the consumption of time within the budget constraint through the generalized cost of commuting. A reasonable questioning may be this is an unrealistic assumption, due to people generally do not take into account the time spent within their monetary expenditures. Also, walking and commuting times (in presence of public transportation stations) generate different trade-offs than those in models with only private cars. For these reasons, in this section we show an extension of our basic model considering leisure time in the utility function, and adding a time constraint in the consumer maximization problem.

In this extended model, residents are homogeneous in preferences, which are represented by a common strictly quasi-concave utility function $U(c, q, l)$ where l is leisure time. Residents work the same number of hours b and commute t hours per day (each day has T hours) depending on their housing location with respect to the CBD. Residents earn the same wage rate w, thus, they are also homogeneous in income. The income wb is spent on paying the fare of public transportation, the consumption of non-housing goods and the rental price of housing. On the other hand, residents spend time on working, commuting, and leisure.

Using the already known budget constraint $wb = c + pq + e$ and the new time constraint $T = b + l + t$, the requirement that the maximized utility level equals u can be written as,

$$\max_{\{q,l\}} V(w(T - l - t) - pq - e, q, l) = u \tag{22}$$

Now two first-order conditions can be obtained from equation (22). The first one is known (equation (2)), where the marginal substitution rate between the marginal utility of q and c equals p. The second one is shown in the equation (23), where the marginal substitution rate between the marginal utility of l and c is obtained, which equals w.

$$\frac{V_l(w(T - l - t) - pq - e, q, l)}{V_c(w(T - l - t) - pq - e, q, l)} = w \tag{23}$$

The spatial equilibrium condition shown above in the equation (3) is also considered. As we did above, now these three conditions allow us to obtain solutions for p and q through totally differentiating them with respect to x. These variables will give us an idea of urban form along the city. The main results are shown in equations (24) and (25).

$$\frac{\partial p}{\partial x} = -\frac{1}{q}\left(w\frac{\partial t}{\partial x} + \frac{\partial e}{\partial x}\right) = \begin{cases} ks < x < \bar{x}^k \rightarrow \frac{\partial t}{\partial x} > 0 \rightarrow \frac{\partial p}{\partial x} < 0 \\ \bar{x}^k < x < (k+1)s \rightarrow \frac{\partial t}{\partial x} < 0 \rightarrow \frac{\partial p}{\partial x} > 0 \end{cases} \tag{24}$$

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As we can see, the result derived in the equation (24) is similar to the equation (12). While the partial derivative of e with respect to x equals zero, the partial derivative of t with respect to x depends on the location regarding to $\bar{x}^k$ between stations. That is to say that the rental price of housing is higher near stations, which is the same intuition acquired in the basic model. The main difference lies in the value of time. While in the basic model the value of time θ is an exogenous parameter, this extension allow us to value time as the wage rate.

$$\frac{\partial q}{\partial x} = \left[\frac{\partial \left(\frac{V_q}{V_c}\right)}{\partial q} - \frac{\frac{\partial \left(\frac{V_q}{V_c}\right)}{\partial l} \frac{\partial \left(\frac{V_l}{V_c}\right)}{\partial q}}{\frac{\partial \left(\frac{V_l}{V_c}\right)}{\partial l}} \right]^{-1} \frac{\partial p}{\partial x} = \eta \frac{\partial p}{\partial x} = \begin{cases} ks < x < \bar{x}^k \to \frac{\partial p}{\partial x} < 0 \to \frac{\partial q}{\partial x} > 0 \\ \bar{x}^k < x < (k+1)s \to \frac{\partial p}{\partial x} > 0 \to \frac{\partial q}{\partial x} < 0 \end{cases} \quad (25)$$

In the equation (25) the term η is more complex than in the basic model because it is composed by four different partial derivatives. The partial derivative of V_q/V_c with respect to q is negative, because the marginal utility of q divided by the marginal utility of c decreases when q increases. The same intuition can be used for the partial derivative of V_l/V_c with respect to l. The cross partial derivatives of V_q/V_c with respect to l and of V_l/V_q with respect to q yield positive values close to zero. That is, the negative sign of the partial derivative of V_q/V_c with respect to q dominates the expression and makes η to be negative. This sign of η equals the one obtained in the previous section and allow us to conclude that the size of dwellings are larger near $\bar{x}^k$ and smaller near stations.

In summary –and considering these results as a proof– the effect of public transportation on the urban form of cities remains identical, with respect to the basic model, when the model includes variables of time.

4.2. Considering heterogeneous consumers

The basic model presented in section 2 can be extended to study the effect of public transportation on the location of income groups along the city. As we mentioned in the Introduction, this approach has already been studied by authors, such as Alonso (1964) or Muth (1969), in a model where only private cars are available to commute. Their spatial equilibrium generates poor and rich neighborhoods with perfect segmentation. Particularly, assuming that income elasticity of demand for land is greater than the income elasticity of commuting costs (or of the value of time), poor residents will live in city centers. According to Becker (1965), this happens if the income elasticity of demand for land is greater than one. Nevertheless, the evidence does not support income segregation across space. For instance, Persky (1990) found evidence of income mixing in both the center and the suburbs in Chicago. Glaeser et al. (2008) showed that household income is lower close to the CBD, then rises at the boundary between the inner city and the suburbs, and then falls again. That is, they report a U-shaped curves plotting incomes along the city. The same authors (Glaeser et al., 2008) estimates that public transportation is two to three times more important than the income elasticity of demand for land in explaining the central location of the poor households. In summary, the related literature has empirically found a different pattern than the classic model predicts and some previous evidence indicates public transportation may have a relevant role in the location behavior of some residents.

We introduce income heterogeneity in the same way as the mentioned literature, by assuming that consumers earn different wage rates and work the same number of hours. For simplicity –and without loss of generality– we consider two different income groups: poor and rich consumers. Residents are homogeneous in their preferences, which is represented by the same common strictly quasi-concave

utility function $U_j(c_j, q_j)$. Nevertheless, consumers are heterogeneous in wage rates w_j. While in the previous basic model the urban equilibrium must yield identical utility level for all individuals, in this model the poor and rich groups reach different utility levels u_j. Poor consumers are denoted by $j = P$ and rich consumers are denoted by $j = R$.

The budget constraint is the same as the previous basic model. Residents, both rich and poor, consume non-housing goods c_j, housing floor space q_j and spend a generalized cost of commuting on public transportation ρ_j. Therefore, using the constraint $w_j b = c_j + p_j q_j + \rho_j$, the consumer optimization problem can be written as follows,

$$\max_{\{q\}} V(w_j b - p_j q_j - \rho_j, q_j) = u_j \tag{26}$$

Both the first-order condition and the spatial equilibrium condition are analogous to those in equations (2) and (3), but now considering the sub index j in order to point rich and poor residents.

4.2.1. Public transportation modelling

The assumptions described in the section 2.3 are subtly modified. For simplicity, we assume all residents have the same work arrival time and they commute on the same public transportation vehicle. As we are modelling two income groups, there is an indifference point of walking upstream or downstream for each group that is not necessarily at the same position. We denote the indifference location between the ks and the $(k + 1)s$ stations as $\bar{x}_j^k$, which may (or may not) be different for rich and poor consumers.

As we did above, first we can analyze the simplest case where residents live between the first station and the CBD. The location of $\bar{x}_j^0$ is obtained from the equation where the generalized cost of walking from $\bar{x}_j^0$ to the first station and then riding in public transportation equals the generalized cost of walking from $\bar{x}_j^0$ to the CBD. The indifference location $\bar{x}_R^0$ and $\bar{x}_P^0$ is determined by the equations (27) and (28).

$$e^s + \theta^R \left(\frac{s}{v_f} + \frac{s - \bar{x}_R^0}{v_c} \right) = \theta^R \frac{\bar{x}_R^0}{v_c} \tag{27}$$

$$e^s + \theta^P \left(\frac{s}{v_f} + \frac{s - \bar{x}_P^0}{v_c} \right) = \theta^P \frac{\bar{x}_P^0}{v_c} \tag{28}$$

In contrast to the section 2.3., we are not only interested in describing the location of $\bar{x}_R^0$ and $\bar{x}_P^0$, but also in the relative position between them. This can be obtained subtracting the equation (28) from the equation (27), result which is shown in the equation (29).

$$\bar{x}_R^0 - \bar{x}_P^0 = \frac{v_c}{2} e^s \left(\frac{1}{\theta^R} - \frac{1}{\theta^P} \right) \tag{29}$$

As we can observe from the equation (29), the relative location of $\bar{x}_R^0$ with respect to $\bar{x}_P^0$ depends on the monetary cost of time, represented by θ^R and θ^P. On one hand, if the value of time is the same for rich and poor residents, both $\bar{x}_R^0$ and $\bar{x}_P^0$ will be located at the same distance x from the CBD. On the other hand, if the value of time for rich residents is higher than for poor residents, the equation (29) will take a negative value. That is, $\bar{x}_R^0$ will be located to the left of $\bar{x}_P^0$. This is what we assume henceforth, that the value of time is higher for the rich ($\theta^R > \theta^P$).

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This analysis can be easily generalized for neighborhoods between stations along the city. The additional consideration that must be taken into account is that the number of residents who will board the public transportation at the upstream station and at the downstream station is different. This issue is more complex than before because it may happen that one income group begins to walk to the upstream station while the other group walks to the downstream station.

Solving the equation (30) and (31) allows us to obtain $\bar{x}_R^k$ and $\bar{x}_P^k$, respectively.

$$e^{ks} + \theta^R\left(\frac{ks}{v_f} + t_b \int_{\bar{x}_R^0}^{\max \bar{x}_j^{k-1}} xDdx + \theta^R \frac{\bar{x}_R^k - ks}{v_c}\right) = e^{(k+1)s} + \theta^R\left(\frac{(k+1)s}{v_f} + t_b \int_{\bar{x}_R^0}^{\max \bar{x}_j^k} xDdx + \theta^R \frac{(k+1)s - \bar{x}_R^k}{v_c}\right) \tag{30}$$

$$e^{ks} + \theta^P\left(\frac{ks}{v_f} + t_b \int_{\bar{x}_P^0}^{\max \bar{x}_j^{k-1}} xDdx + \theta^P \frac{\bar{x}_P^k - ks}{v_c}\right) = e^{(k+1)s} + \theta^P\left(\frac{(k+1)s}{v_f} + t_b \int_{\bar{x}_P^0}^{\max \bar{x}_j^k} xDdx + \theta^P \frac{(k+1)s - \bar{x}_P^k}{v_c}\right) \tag{31}$$

Using the same procedure as before, the relative location of $\bar{x}_R^k$ with respect to $\bar{x}_P^k$ can be obtained subtracting the equation (31) from the equation (30). This result is shown in the equation (32).

$$\bar{x}_R^k - \bar{x}_P^k = \frac{v_c}{2}\left(e^{(k+1)s} - e^{ks}\right)\left(\frac{1}{\theta^R} - \frac{1}{\theta^P}\right) \tag{32}$$

The main difference between equations (29) and (32) is the term associated with the fare of public transportation at consecutive stations ($(k+1)$ and k). Particularly, this term indicates that the relative location of $\bar{x}_R^k$ with respect to $\bar{x}_P^k$ depends also on the fare policy in place. Regardless of the relative value of the monetary cost of time for rich and poor people, if the public transportation has a flat fare along the city, $\bar{x}_R^k$ and $\bar{x}_P^k$ will locate at the same distance x from the CBD. In any other case, the relative location will depend on both terms, that is, on the assumption about the value of time and on the modelling decision about the fare policy along the city. As we assume that $\theta^R > \theta^P$ holds, there are two cases. An increasing fare ($e^{(k+1)s} > e^{ks}$) when distance x increases from the CBD will imply $\bar{x}_R^k < \bar{x}_P^k$. That is, we obtain the same result as before, between the CBD and the first station, because the fare of public transportation at the CBD is zero. On the contrary, a decreasing fare ($e^{(k+1)s} < e^{ks}$) will imply the opposite, that is, $\bar{x}_R^k > \bar{x}_P^k$.

4.2.2. Location of income groups along the city

As there is a fixed number of poor and rich, and housing is essential (everyone must consume a positive amount), there must be locations where the rich outbid the poor and locations where the opposite is true. At a certain location, the group with the higher willingness to pay for housing floor space – represented by p_j– will in equilibrium live in that location. Consider that there are points that separate income groups and denote one of those points $\hat{x}$. At $\hat{x}$, rich and poor consumers are willing to pay the same for square meters of housing, which implies that the rich residents' consume more housing than the poor residents' (i.e. $q_R > q_P$), assuming it is a normal good. Using the condition in equation (12), we can relate the slopes of the bid rent functions.

As we showed in the section 2.4., the partial derivative of p with respect to x is given by the equation (12). This result is straightforward to generalize for both income groups as is shown in the equation (33).

$$\frac{\partial p_j}{\partial x} = -\frac{1}{q_j}\frac{\partial \rho_j}{\partial x} \qquad (33)$$

Let us first analyze the case where the public transport fare is flat ($\partial e/\partial x = 0$). Consider the neighborhood between the CBD and the first station. For each income group j the bid rent gradient ($\partial p_j/\partial x$) changes sign at $\bar{x}_j^0$, so it is a convex function that is downward sloping between the CBD and $\bar{x}_j^0$ and upward sloping between $\bar{x}_j^0$ and s (the location of the first station). Assume that the bid rent functions cross at some point in this interval. As both are convex functions, they could cross once or twice. At any of the crossing points, the difference in slopes of the rent gradients is given by:

$$\frac{\partial p_R}{\partial x} - \frac{\partial p_P}{\partial x} = \frac{1}{q_P}\frac{\partial \rho_P}{\partial x} - \frac{1}{q_R}\frac{\partial \rho_R}{\partial x} \qquad (34)$$

If the first crossing point is such that $CBD < \hat{x} < \bar{x}_R^0$, both groups are walking to the CBD and equation (34) can be rewritten as:

$$\frac{\partial p_R}{\partial x} - \frac{\partial p_P}{\partial x} = \frac{1}{v_c} \cdot \left[\frac{\theta^P}{q_P} - \frac{\theta^P}{q_R}\right] \qquad (35)$$

We know both p_R and p_P have a negative slope and the result of the equation (35) is analogous to the one in the standard model. The group with the highest ratio of commuting costs per unit distance to housing consumption will have the steepest bid curve and will live next to the CBD (from 0 to $\hat{x}$). In other words, if the income elasticity of commuting costs is smaller than the income elasticity of the demand for housing, the poor will live next to the CBD and if the opposite holds, the rich will live close to the CBD. As both curves change sign at different locations they may cross again.

For simplicity of exposition we will now focus on the case in which the poor have a highest ratio of commuting costs per unit distance to housing consumption and hence live next to the CBD. The opposite case is analogous. So, poor residents live between the CBD and $\hat{x}$ and rich residents live from $\hat{x}$. If the curves do not cross again before s, the rich will live from $\hat{x}$ onwards. The most interesting case, and the novelty of this approach, is when the bid curves do cross again. In this case, we have a second point of separation of income groups. The pattern of residential sorting between the CBD and the first station is thus a group of poor people living next to the CBD and next to the station with a group of rich people in between. Figure 5 shows the bid curves of the rich (dashed lines) and of the poor (solid lines), and two possible combinations of sorting. What happens after the first station is intuitive, the curves cross again at some point and they may cross several times as well. Therefore this poor-rich-poor pattern may hold in many pairs of stations before the rich always outbid the poor in the suburbs.
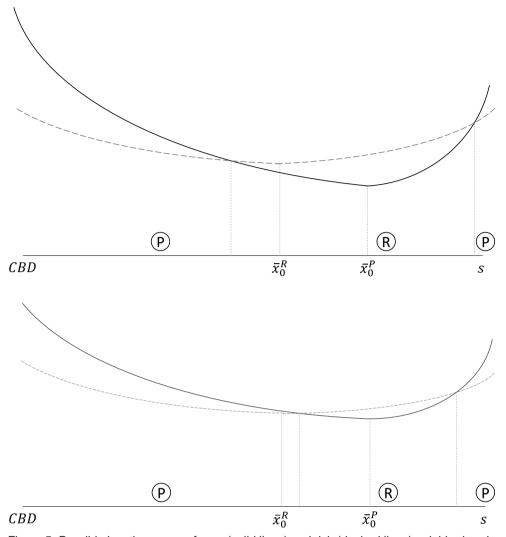
Figure 5: Possible location cases of poor (solid lines) and rich (dashed lines) neighborhoods

As we mentioned at the beginning, Alonso's theory indicates that wealthier consumers will be willing to bid more for peripheral sites than the poor, thus, his equilibrium shows that income increases with greater distance from the CBD. Nevertheless, these conclusions change when considering public transportation. Our results show that in presence of public transportation, high and low income groups may now be mixed along the city. Particularly, between stations, we can observe two possible cases and up to three neighborhoods (see Figure 5). Despite the above, the general configuration is only one: while poor residents tend to live near stations, rich residents tend to live at the middle between stations. Due to the discontinuous effect in commuting times, yielded by walking to public transportation stations, there would now be a mix of poor and rich neighborhoods that intertwine along the city, a pattern that can be observed in reality and has been reported in the literature (Persky, 1990; Glaeser et al. (2000)).

5. Conclusions

This paper provides insights about how the urban form may be influenced by the public transportation system and its characteristics. Based on some basic assumptions of Alonso's spatial model, we show that urban form predicted for cities with only private cars is different when residents ride on public transportation. Our results show that variables such as prices, dwelling sizes and heights of buildings

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do not follow a monotonic behavior along the city, but they do between stations. For example, the rental price of housing and the land rent are highest at stations than in between stations, but these peaks are decreasing with distance to the CBD. At the same time, the tallest buildings are located near stations. Our model also provides novelties regarding to the location of different income groups along the city. While Alonso's model foretells a city with perfect segregation between rich and poor groups, this model shows that the presence of public transportation may explain that high and low income neighborhoods intertwine along the city.

From this research, some additional questions can also be set. The model allows us to find how different fare policies of public transportation (e.g. plain fare, fare depending on distance or fare depending on zones) and the spacing between stations influence the urban form through the impact on p, q, r and S. Also, we can develop a comparative static analysis adding equilibrium conditions related to the city boundary $\bar{x}$ (the location where the urban land rent r equals the agricultural rent r_a) and the population P (it must fit inside the city boundary $\bar{x}$). For instance, this analysis would allow us to understand how the size of the city changes when fares of public transportation e or wage rates w vary.

Finally, the model can be extended to include both, private cars and public transportation, jointly. From this model similar questions arise, such as the change in the size of the city or the form of relevant variables (p, q, r and S), but now in a more complete framework. More importantly, this extended model will allow us to analyze the impact of more realistic transportation policies on urban form. Road congestion pricing, public transportation subsidies and dedicated bus lanes can be modeled, both alone and together, in order to understand its impact in the wider economy and structure of the city. Importantly, we will also be able to study how the city evolves in terms of urban sprawl, distribution of dwellings, among others, under different set of (transportation) policies.

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