A COMPLEMENTARITY FORMULATION FOR THE SYSTEM OPTIMAL DYNAMIC TRAFFIC ASSIGNMENT PROBLEM EMBEDDED WITH A NODE-BASED FLOW PROPAGATION MODEL

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Abstract

This paper proposes a variational-inequality-based framework for modeling dynamic traffic equilibrium (DTA) assignment problems. The framework can support any type of travel choice defined over a finite set of alternatives and it does not explicitly assume a network loading model. It also does not necessitate path enumeration, which has been a major source of computational burden in earlier models. We provide a simple complementarity formulation for such a general equilibrium assignment problem, derive its equivalent variational inequality formulation and describe a projection algorithm for solving it efficiently. Under some mild conditions on the network loading models, the algorithm is shown to converge subsequentially. The practical implications of such convergence are examined by applying the framework to solve the system-optimum (SO) DTA problem with two standard networks. A further discussion on the efficiency of the algorithm is also presented.

Keywords: dynamic traffic assignment, complementarity formulation, variational inequality, user equilibrium, system optimal.

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1. INTRODUCTION

Over the last few decades, several variants of the dynamic traffic assignment (DTA) problem have been proposed based on a wide-ranging modeling considerations. Many of the earlier works dealt with a subvariety called system-optimum (SO) problems. As the name indicates, their goal was to assign road users to paths to minimize their total travel cost. This notion is quite unrealistic, however, as no such central authority assigning road users to paths exists in reality. Instead the users choose their paths based on their personal costs, and they usually do not consider and may not even know the travel costs of their neighbors. This gives us a more realistic alternative of user equilibrium (UE), which can be defined as the situation where no user can improve their cost by unilaterally changing their paths. A further stochastic modification to this notion gives us stochastic user equilibrium (SUE), where each user chooses a path that minimizes their *perceived* travel cost, which is modeled as a random variable. Each of the DTA problems studied so far in the literature can be broadly categorized with these three notions. Despite its lack of realism, the SO-DTA problem still retains its importance in the literature for benchmarking the efficacy of traffic management strategies. It is also used as a subproblem when solving more advanced problems like network design and congestion pricing.

Furthermore, a dynamic network loading model, that describes the traffic flow dynamics, forms a core component of any DTA model. Some of the recent works have used cell transmission models (CTMs) as implicit simulation methods and avoid modeling artifacts such as holding back of the traffic at junctions despite the availability of space downstream (Doan and Ukkusuri, 2012). In Szeto and Lo (2004), both departure time and route choices are incorporated in the model for a general network with elastic demand. Similarly, Han et al. (2011) formulated the DUE problem within the framework of complementarity theory for a network with a single origin-destination (O-D) pair and multiple user classes with elastic demand. This work is extended for multiple O-D pairs by Ukkusuri et al. (2012) with maximum and average travel time models and they also provide a solution procedure based on the projection algorithm from the VI theory. Doan and Ukkusuri (2015) developed a similar DSO formulation for general multi-OD networks with both route and departure time choices and they also provided an estimation procedure for path marginal costs that accurately captures the effect of any perturbation on all other traffic. However, in these works, the users' choices are restricted to a fixed number of paths determined during the preprocessing step of the algorithm. Therefore, having a small number of these paths could lead to suboptimal solutions, whereas a larger number would require high computational resources. Gentile (2016) addresses this issue by extending the Gradient Projection algorithm for a UE problem with sequential local choices at nodes directed towards a destination. However, the algorithm was described in the paper as "only heuristics with no guarantee of convergence."

The present work extends upon these recent complementarity theory and variational-inequality-based works and propose a flexible framework for modeling traffic equilibrium assignment models. This refers not only UE problems, as SO and SUE problems can also be formulated as an equilibrium problem with the right choice of the cost function (Gentile, 2018). Along with a detailed description of the framework and a projection algorithm for solving the problem, a proof of solution existence and the subsequential convergence of the algorithm is also presented. The framework is applied to solve the SO-DTA problem on two standard networks and discuss the results.

2. NOTATION

The notation used in this paper is briefly described in this section. It should be especially noted that a couple of symbols (h and H) are used both as indices

h	Index of a choice variable
H	Set of all indices of the choice variables corresponding to a particular location. It is also interpreted as an index for the location when used as a subscript.
$\mathbf{H}$	Solution space of the DTA problem
$\mathcal{H}$	Set of all sets of choice variables indices (H's) corresponding to different choice locations in the network
x_h	Value of the choice variable h
c_h	Cost associated with the choice variable h
c_H	Minimum cost associated with a choice location H
$\mathcal{C}$	Vector-valued function mapped from the solution space $\mathbf{H}$ to $\mathbb{R}^{\lvert\mathbf{H}\rvert}$

The above set of symbols form the core notation used in this work. Some locally-defined symbols are further used for some general definitions and results.

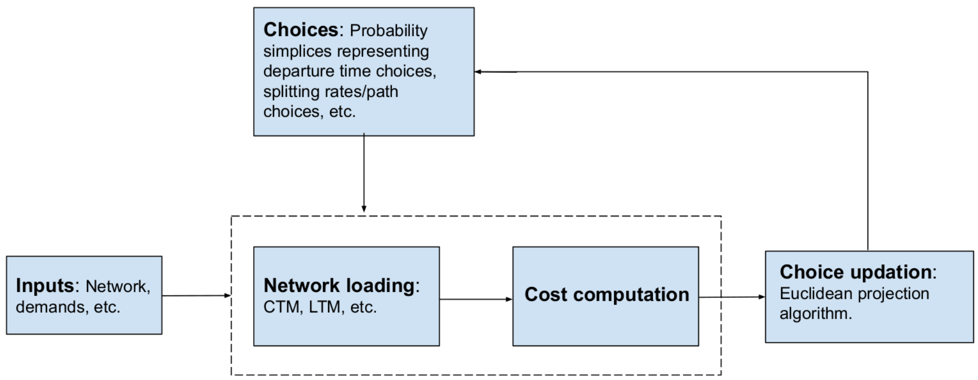


Figure 1: Outline of the DTA framework

3. DESCRIPTION OF THE PROPOSED FRAMEWORK

Figure 1 broadly outlines the various elements of the proposed framework. In this section, we describe the conditions needed to be satisfied by some of the elements to guarantee some important theoretical results.

3.1. Choices

As stated previously, the proposed framework limits the travel choices to those defined over a finite set of alternatives. These travel choices are assumed to be made by the users locally and sequentially, as described by Gentile (2016). For example, the users choose the exit link from a junction, if and when they reach it during the simulation. Thus, each of the choices at the "choice locations" are represented by a finite set of variables that form a probability simplex, i.e. they satisfy the following equations:

$$\begin{aligned} x_h &\geq 0,\ h \in H \\ \sum_{h \in H} x_h &= 1 \end{aligned} \tag{1}$$

where x_h and H represent the value of a choice variable h and the set of all choice variable indices corresponding to a particular choice location. And we assume that $|H| < \infty$.

Note that the path-based models can also be formulated within this framework as a special case, where the users make a "local" choice at the point of their entry among the enumerated paths. Now, the following condition on the solution space of the problem holds true.

Condition 1. *The solution space* ***H*** *of the problem can be obtained as a cartesian product of a finite number of probability simplices.*

In the rest of the paper, we denote by **H** and $\mathcal{H}$, the solution space and the set of all the probability simplices forming it, respectively.

Proposition 3.2. ***H*** *is non-empty, compact and convex set.*

The proof follows by noting the inheritence of these properties from the probability simplices. Later on in this paper, we will need another property of **H** stated below.

Proposition 3.3. *Every point $x \in$* ***H*** *satisfies Abadie's constraint qualification (ACQ).*

The proof of the proposition is given in Appendix B.

3.4. Complementarity formulation of the DTA problem

In a user-equilibrium (UE) assignment, no road user can lower their travel cost by unilaterally changing their travel choices. Wardrop's equilibrium principle further states that the costs associated with the alternatives actually chosen by the users are equal and minimal. This can be stated mathematically as follows:

$$(c_h - c_H) \times x_h = 0, \quad \forall h \in H, \forall H \in \mathcal{H}, \tag{2}$$

where $x(\in \mathbf{H})$ is the UE assignment solution; h, c_h and c_H are the choice variable index, its associated cost and the minimum cost among the alternatives in $H(\ni h)$. Note that c_h denotes a function of $x \in \mathbf{H}$, but the $(.)$ notation is dropped in this paper to avoid visual clutter.

By treating c_h as marginal travel cost associated with the choice h, the system-optimal problem also results in eq. 2. Indeed, Gentile (2018) shows that even a stochastic user equilibrium problem with multinomial logit route choices has a function that can be taken as c_h and obtain eq. 2. With the additional assumption of boundedness on c_h, we can derive a complementarity formulation from eq. 2 and eq. 1.

Proposition 3.5. *Suppose that there exists a constant* $B \geq 0$ *such that* $c_h(x)+B > 0, \forall x \in \boldsymbol{H}, \forall h \in H \in \mathcal{H}$. *Then, from 1 & 2 can be combined to give the following:*

$$\begin{aligned} &0 \leq x_h \perp \hat{c}_h - \hat{c}_H \geq 0, \; \forall\, h \in H, \; \forall H \in \mathcal{H} \\ &0 \leq \hat{c}_H \perp \sum_{h \in H} x_h - 1 \geq 0, \; \forall H \in \mathcal{H} \end{aligned} \tag{3}$$

where $\hat{c}_h = c_h + B$ *and* $\hat{c}_H = c_H + B$.

Proof. From the boundedness assumption, the right-hand-side of eq. 3 must be equal to 0. This gives us the original equations back. □

The assumption of boundedness is easily justifiable because we are concerned with time-discretized models defined over a finite observation period. Without loss of generality, we will assume that $B = 0$ for the rest of the paper.

3.6. Costs

Travel time costs and schedule delay costs are two most commonly considered types of costs. Algorithms based on dynamic programming technique can be used to compute the travel cost (c_h) associated with each choice variable h. Average travel costs can be used for computing a UE solution, whereas the solving an SO problem requires marginal costs. Depending on the model considerations, alternative cost functions can be used, as long as they satisfy the following condition.

Condition 2. *The cost (*c_h*) associated with each choice variable* h *is a set-valued map that is:*

a) *non-empty,*
b) *contractible,*
c) *compact-valued and*
d) *upper semi-continuous.*

Ukkusuri et al. (2012) gives rigorous definitions of these terms, which are reproduced here in Appendix C. It should also be noted here that c_h is a function of a point x in the solution space $\mathbf{H}$, but the $(.)$ notation is dropped for visual simplicity.

So, if c_h can be given as a continuous function of $x (\in \mathbf{H})$, it is easy to see that all the above conditions are satisfied. However, there are some plausible cost functions, for eg. marginal costs, which are often discontinuous. In some of the popular CTM-based network loading models, the discontinuity results from the presence of $min(.)$ function. The above conditions can be shown to be satisfied in the presence of this function.

3.7. Generalized Variational Inequality (GVI) formulation

Definition 1. *(GVI) Given a non-empty convex closed set $K \subseteq \mathbb{R}^n$ and a set-valued map $F : K \rightrightarrows \mathbb{R}^n$, a GVI is to find a vector $x \in K$ and $z \in F(x)$ such that*

$$z^T(y - x) \geq 0, \ \forall y \in K.$$

A couple of results from GVI theory relevant for what follows are given below.

Lemma 3.7.1. *If K is a compact set, and F is non-empty, contractible, compact-valued and upper semi-continuous, then $GVI(K, F)$ has a solution.*

Proposition 3.8. *Facchinei and Pang (2003, Proposition 1.3.4). Let $K \equiv \{x \in \mathbb{R}^n : h(x) = 0, g(x) \geq 0\}$, with $h : \mathbb{R}^n \to \mathbb{R}^l$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ being vector-valued continuously differentiable functions. Let F be a mapping from K into $\mathbb{R}^n$. The following two statements are valid:*

(a) Let $x \in SOL(K, F)$. If Abadie's CQ holds at x, then there exist vectors $\mu \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^n$ such that

$$\begin{gathered} 0 = F(x) + \sum_{j=1}^{l} \mu_j \nabla h_j(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) \\ 0 = h(x) \\ 0 \leq \lambda \perp g(x) \leq 0. \end{gathered} \tag{4}$$

(b) Conversely, if each function h_j is affine and each function g_i is convex, and if (x, μ, λ) satisfies (4), then x solves the $VI(K, F)$.

Now, let us denote by $\mathcal{C}(\mathbf{h})$, a set-valued map from $\mathbf{H}$ to $\mathbb{R}^{|\mathbf{H}|}$, that gives us the cost values corresponding to each of the choice variables.

Proposition 3.9. *$(x^*, \boldsymbol{C}^*)$ is a solution of 3 if and only if it is a solution of $GVI(\boldsymbol{H}, \mathcal{C})$.*

Proof. Let x^* be a solution of $VI(\mathbf{H}, \mathcal{C})$. By the KKT conditions of the VI given in proposition 3.8, there exists $\mu^* = \{\mu^*_H, \ \forall \, H\}$ and $\lambda^* = \{\lambda^*_h, \ \forall \, h \in H, \ \forall \, H\}$ such that,

$$\begin{aligned} & c_h(x^*) + \mu_H - \lambda_h = 0, \forall \, h \in H \in \mathcal{H}, \\ & \sum_{h \in H} x_h = 1, \forall \, H, \\ & 0 \leq x_h \perp \lambda_h \geq 0, \forall \, h \in H \in \mathcal{H}. \end{aligned} \tag{5}$$

Using Proposition 3.5 and rearranging the above equations, we obtain the complementarity formulation 3. The necessity condition can also be proven with a similar logic.

□

The solution existence result now follows.

Theorem 3.9.1. *The complementarity formulation 3 has a solution.*

Based on conditions 1 & 2, it can be seen that lemma 3.7.1 is applicable for $GVI(\mathbf{H}, \mathcal{C})$. And from proposition 3.9, the above theorem follows.

4. SOLUTION ALGORITHM

A classic VI approach called "projection method" can be adapted to solve the DTA models with the proposed framework. Given an initial solution x^0 for a $GVI(K, F)$, the method computes a sequence of points $\{x^k\}_{k=1}^{\infty}$ as follows:

$$x^{k+1} = \Pi_K(x^k - F(x^k)),\ \forall\, k = 1, 2, \ldots$$

where $\Pi_K(x)$ is the projection of x on K, i.e.,

$$\Pi_K(x) = \arg\min_{z \in K} ||z - x||^2.$$

To prove the convergence of this method, the map F must be "monotonous". In the case of DTA problems, the cost functions generally do not satisfy this property. But since our solution space is compact, we can still have "subsequential convergence".

Proposition 4.1. *Let* $x^0 \in \boldsymbol{H}$ *and* $x^{k+1} = \Pi_{\boldsymbol{H}}(x^k - \mathcal{C}(x^k)), \forall k = 1, 2, \ldots$. *Then, the following statements hold.*

(a) *There exists a convergent subsequence of* $\{x^k\}_{k=0}^{\infty}$.
(b) *For any convergent subsequence of* $\{x^k\}_{k=0}^{\infty}$, *the limiting point* x^* *is a solution of 3.*

The proof follows from the fact that any sequence in a compact set must have a convergent subsequence.

Now the complete solution algorithm can be described as follows:

Step 1. Initialize a feasible choice vector x.
Step 2. Perform the simulation with a network loading model and compute the costs associated with each choice $\mathcal{C}(x)$.
Step 3. For each choice set H, update its variables as follows: $x_H := \min_{\mathbf{z} \in \mathbf{H}_H} ||\mathbf{z} - \mathcal{C}_H(x)||$.
Step 4. Terminate the algorithm if x satisfies some predecided criteria. Otherwise, go to Step 2.

Note that Step 3 above can be performed parallelly for the sets of choice variables corresponding to choice locations. Since each of those sets define a probability simplex, a very efficient algorithm described in Wang and Carreira-Perpinán (2013) can be used for updating the choices.

5. NUMERICAL EXAMPLES

In this section, the proposed framework and algorithm are applied to two standard test networks: the first one (figure 2a), originally used in Ziliaskopoulos (2000), has 10 cells, two diverging sections and a single O-D pair (1-10), and the second one (figure 2b), originally given in Nguyen and Dupuis (1984), has 57 cells, eight diverging intersections and four O-D pairs (1-48; 1-57; 14-48; 14-57). Unlike in the previous studies, no predefined paths are used in the experiments. The cell length in both the cases is 0.5 miles, and the free-flow speed and congestion wave factor of the vehicle type are 30 mph and 0.8, respectively. All the roads have three lanes, and their jam density and flow capacity are 160 vehicles/mile/lane and 24 vehicles/minute/lane, respectively. For simplicity, the same travel cost factors are used for all sinks (α_s=1; β_s=0.5; γ_s=2). For Ziliaskopoulos' network, the target time, maximum departure time and maximum time horizon are 110, 160 and 170, respectively; whereas the same for the Nguyen-Dupuis' network are 60, 80 and 132, respectively.

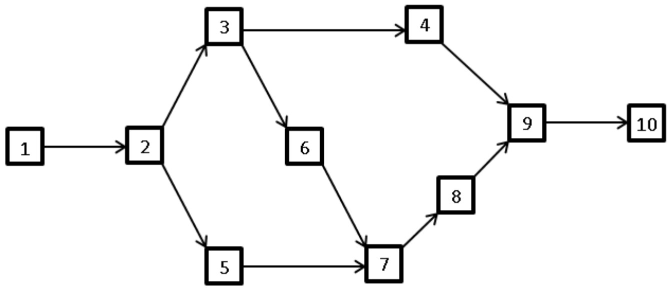
The experiments were conducted with three different demand levels between each O-D pair–600, 1200 and 1800–and different learning rate parameters (0.001, 0.0001). A proportion-based cell transmission model described in Ukkusuri et al. (2012) is used and the path choices are initialized with the shortest paths determined using Dijkstra's algorithm with free-flow travel times. Similarly, all the traffic is departed at the beginning of the simulation.

Table 2: Running times per iteration (in seconds)

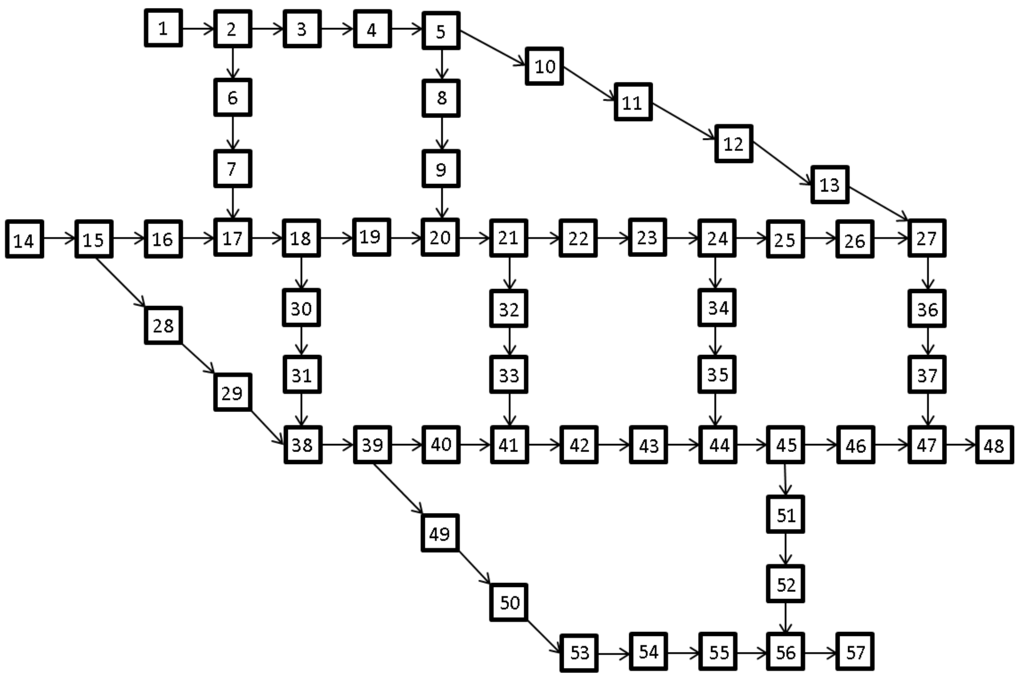
Network	**Node-based**	**Path-based**
Ziliaskopoulos	0.19	0.20
Nguyen-Dupuis	0.85	0.90

Table 2 compares the running times of each iteration of node/path-based SO-DTA problems in both the networks. The simulation code is written in Julia programming language and all the simulations are run on a 64-bit Intel Core i5-8265U 1.60GHz processor. Each experiment was ran for 750 and 300 iterations with Nguyen-Dupuis' and Ziliaskopoulos' networks, respectively. For larger networks, parallelization at various stages of the algorithm can be expected to further improve the running times.

Figures 3, 4 and 5 show the results of the experiments in terms of the convergence of the total system cost. Other convergence measures like mean change in the choices in successive iterations can also be examined but they seem have



(a)



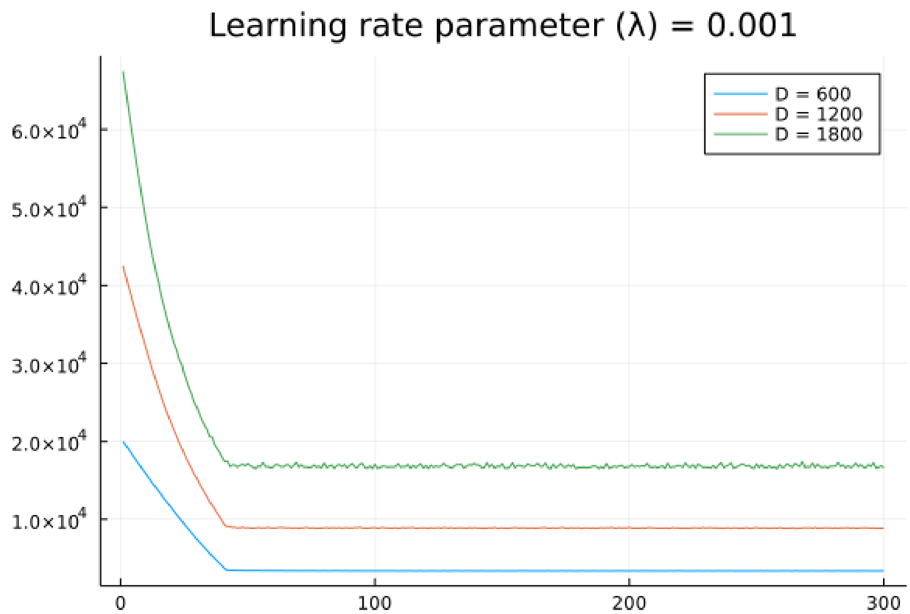
(b)

Figure 2: Test networks: (a) Ziliaskopoulos' network, (b) Nguyen-Dupuis' network

fluctuations with larger magnitudes, which seem to result from a combination of the time-discretization and floating point effects. Note that this is still consistent with our theoretical results, as it only guarantees subsequential convergence.

The following observations can be made from the figures:

1. As expected, problems with larger demands per O-D pair converge to higher total system cost values. Also the smoothness of the convergence seems to be slightly affected for larger demands.
2. Similarly, for smaller learning rate parameter values, the convergence seems to be slower but smoother.
3. Also, it is somewhat surprising that node-based model of Nguyen-Dupuis' network seems to converge to a higher total system cost than its path-based model. But the opposite happens in the case of Ziliaskopoulos' network. We theoretically expect the suboptimality of the path-based models, as we can only select a fixed number of paths. This effect might be countered by variability in the splitting rates of the node-based model over successive iterations. More experiments need to be performed on benchmark networks at different time discretizations to verify this hypothesis.



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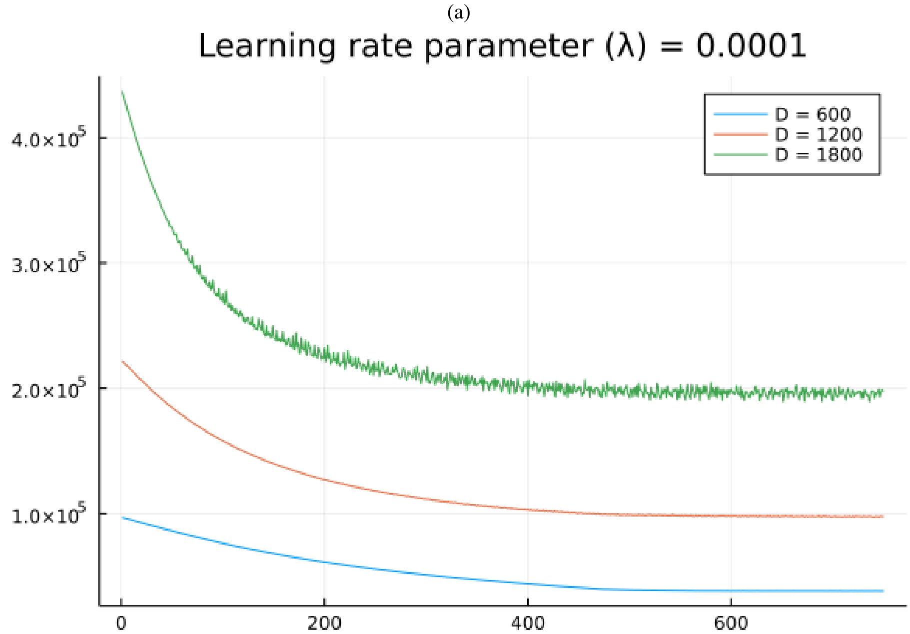
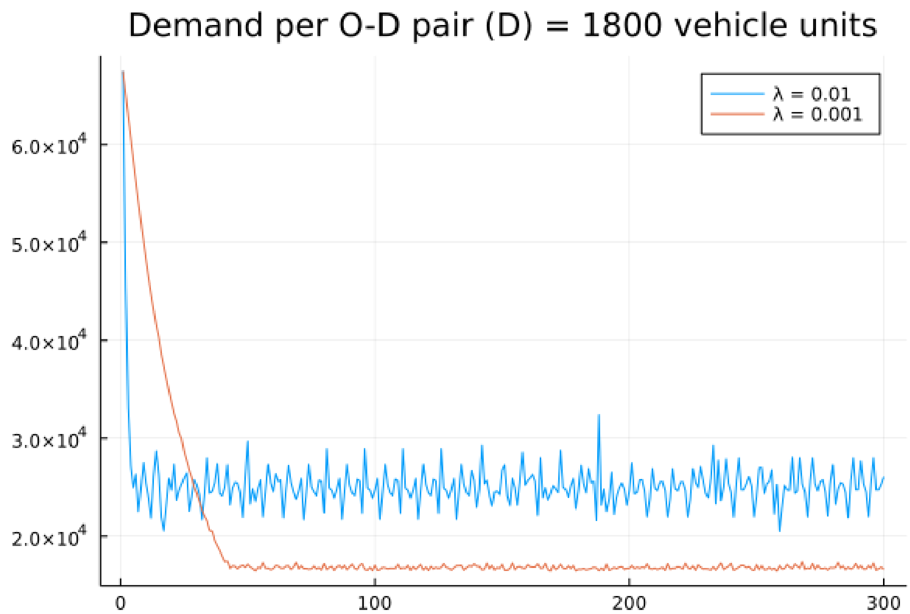


(b)

Figure 3: Convergence of total system cost at different demand levels with the learning rate parameter fixed at 0.0001 on: (a) Ziliaskopoulos' network, (b) Nguyen-Dupuis' network. Note that the X and Y-axes indicate the number of iterations and the total system cost, respectively.



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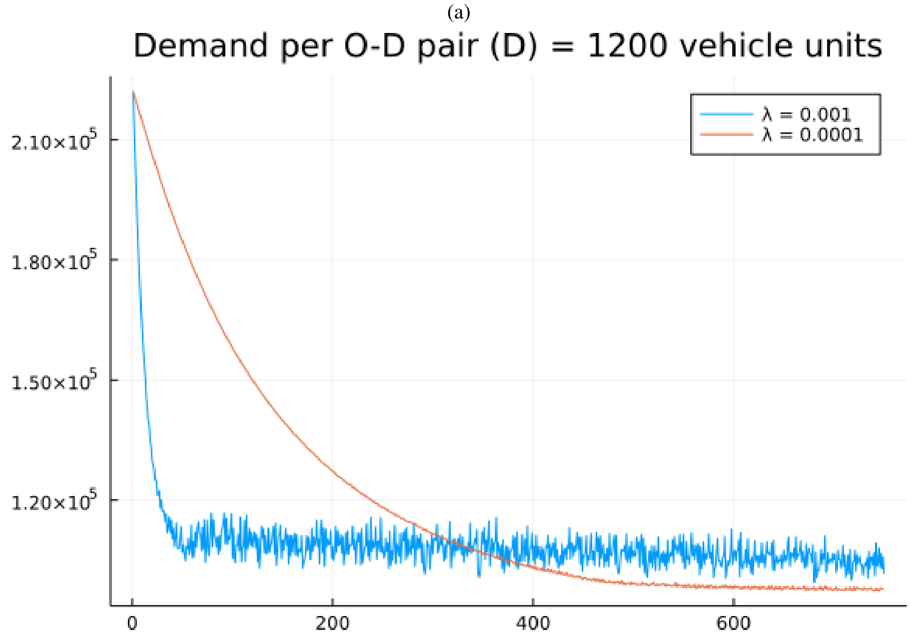
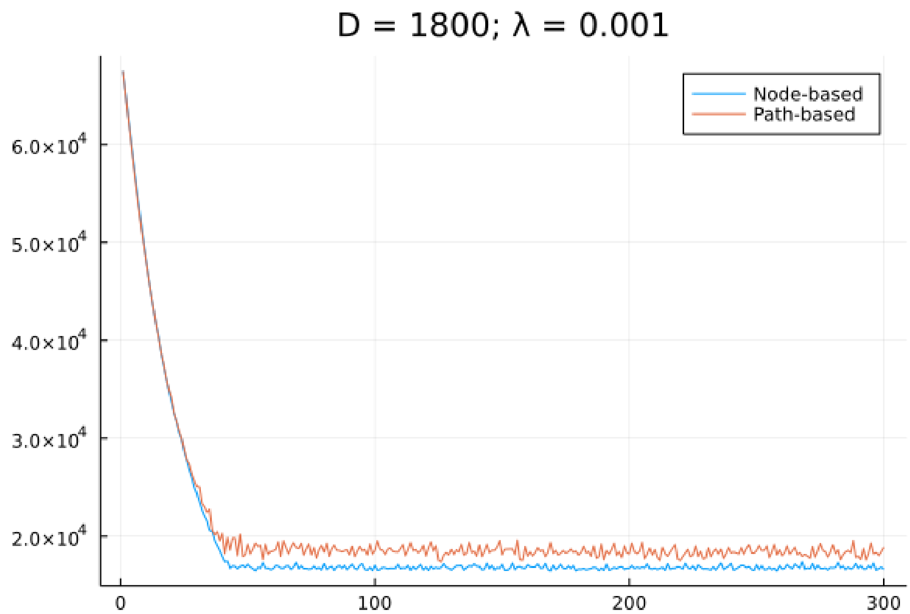
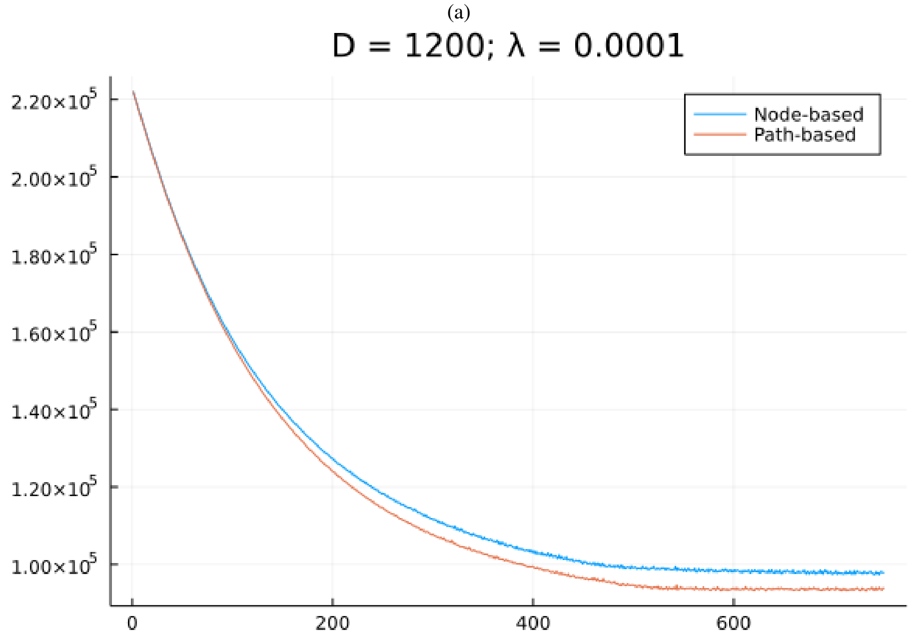


(b)

Figure 4: Convergence of total system cost with different learning rate parameters and a fixed demand per O-D pair: (a) Ziliaskopoulos' network, (b) Nguyen-Dupuis' network. Note that the X and Y-axes indicate the number of iterations and the total system cost, respectively.



(a)



(b)

Figure 5: Convergence of total system cost : (a) Ziliaskopoulos' network, (b) Nguyen-Dupuis' network. Note that the X and Y-axes indicate the number of iterations and the total system cost, respectively.

6. CONCLUSIONS AND FUTURE WORK

The contributions made by this work can be listed as follows:

1. Proposal of a flexible framework to formulate equilibrium-based DTA problems that can accommodate variants of the problem along different dimensions such node/path-based, UE/SO, etc.
2. Proposal of an efficient projection algorithm to solve these problems.
3. A clear description of the conditions needed to be satisfied by the elements of the framework to ensure solution existence and the subsequential convergence of the algorithm.
4. Illustration of the effectiveness of the framework by applying it to solve SO-DTA problems with two standard networks, followed by a discussion of the results.

The framework is general enough to be applicable for a wide-ranging formulations of the DTA problem. On the theoretical front, it seems like the subsequential convergence is the best we can hope for, given the complexity of the problem. The fluctuations observed in the numerical examples seem to result mainly from a combination of the time-discretization and floating point issues. Nevertheless, heuristics can be used to select a suitable assignment after a predecided number of iterations. Similarly, on the practical front, the algorithm has a reasonable running time even without any sort of parallelization. In a future work, we aim to verify its performance with more benchmark networks and other variants of the DTA problem. Furthermore, it can be easily incorporated into more advanced macroscopic applications like network design and congestion pricing.

References

Doan, K. and S. V. Ukkusuri (2012). On the holding-back problem in the cell transmission based dynamic traffic assignment models. *Transportation Research Part B: Methodological 46*(9), 1218–1238.

Doan, K. and S. V. Ukkusuri (2015). Dynamic system optimal model for multi-od traffic networks with an advanced spatial queuing model. *Transportation Research Part C: Emerging Technologies 51*, 41–65.

Facchinei, F. and J.-S. Pang (2003). *Finite-dimensional variational inequalities and complementarity problems*. Springer Science & Business Media.

Gentile, G. (2016). Solving a dynamic user equilibrium model based on splitting rates with gradient projection algorithms. *Transportation Research Part B: Methodological 92*, 120–147.

Gentile, G. (2018). New formulations of the stochastic user equilibrium with logit route choice as an extension of the deterministic model. *Transportation science 52*(6), 1531–1547.

Han, L., S. Ukkusuri, and K. Doan (2011). Complementarity formulations for the cell transmission model based dynamic user equilibrium with departure time choice, elastic demand and user heterogeneity. *Transportation Research Part B: Methodological 45*(10), 1749–1767.

Nguyen, S. and C. Dupuis (1984). An efficient method for computing traffic equilibria in networks with asymmetric transportation costs. *Transportation Science 18*(2), 185–202.

Szeto, W. and H. K. Lo (2004). A cell-based simultaneous route and departure time choice model with elastic demand. *Transportation Research Part B: Methodological 38*(7), 593–612.

Ukkusuri, S. V., L. Han, and K. Doan (2012). Dynamic user equilibrium with a path based cell transmission model for general traffic networks. *Transportation Research Part B: Methodological 46*(10), 1657–1684.

Wang, W. and M. A. Carreira-Perpinán (2013). Projection onto the probability simplex: An efficient algorithm with a simple proof, and an application. *arXiv preprint arXiv:1309.1541*.

Ziliaskopoulos, A. K. (2000). A linear programming model for the single destination system optimum dynamic traffic assignment problem. *Transportation science 34*(1), 37–49.

A. Definition of Abadie's constraint qualification

The definitions given below are reproduced from Facchinei and Pang (2003) for the sake of completeness. Interested readers can refer to the book for more details.

Consider a set $K \subset \mathbb{R}^n$ defined as follows:

$$K := \{x \in \mathbb{R}^n | a(x) = 0, b(x) \leq 0\},$$

where $a : \mathbb{R}^n \to \mathbb{R}^l$ and $b : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable functions.

Definition 2. *(Tangent Cone) For a vector $x \in K$, the tangent cone of K at x, denoted by $\mathcal{T}(x; K)$, is given by:*

$$\{y \in \mathbb{R}^n | \exists \{y^k\} \subset K, \{\tau^k \downarrow 0\} : y^k \to x \text{ and } \frac{y^k - x}{\tau^k} \to y\}. \tag{6}$$

Definition 3. *(Linearization Cone) For a vector $x \in K$, the linearization cone of K at x, denoted by $\mathcal{L}(x; K)$, is given by:*

$$\{v \in \mathbb{R}^n \mid v^T \nabla a_j(x) = 0, \forall j = 1, ..., l \tag{7}$$

$$v^T \nabla b_i(x) \leq 0, \forall i \in \mathcal{I}(x)\}, \tag{8}$$

where $\mathcal{I}(x)$ is the active index set at x, i.e., $\mathcal{I}(x) \equiv \{i : b_i(x) = 0\}$.

Definition 4. *(Abadie's Constraint Qualification) We say that Abadie's constraint qualification holds at any vector $x \in K$ if:*

$$\mathcal{T}(x; K) = \mathcal{L}(x; K).$$

B. Proof of Proposition 3.3

Proof. ACQ is said to be satisfied at a point $x \in \mathbf{H}$ if its tangent cone $\mathcal{T}(x;\mathbf{H})$ is equal to its linearization cone $\mathcal{L}(x;\mathbf{H})$ (see Appendix A). We'll first show that every point $y \in \mathcal{T}(x;\mathbf{H})$ is also in $\mathcal{L}(x;\mathbf{H})$ and then the vice-versa.

Recall from Condition 1 that each set of choice variables $H \in \mathcal{H}$ satisfy eq. 1. These can be rewritten in the standard form given in Appendix A as:

$$\begin{aligned} a(x) &= \sum_{h \in H} x_h - 1 && = 0 \\ b_h(x) &= -x_h && \leq 0,\ h \in H \end{aligned}$$

where x_h is the h-th element of x.

Without loss of generality, we assume for the rest of the proof that $|\mathcal{H}| = 1$. Because of the decoupled nature of the variables, all the equations below can be given separately for each $H \in \mathcal{H}$.

For any $v \in \mathbb{R}^{|H|}$,

$$\begin{aligned} v^T \Delta a(x) &= \sum_{h \in \{1,\ldots,|H|\}} v_h \\ v^T \Delta b_h(x) &= -v_h \end{aligned} \tag{9}$$

where v_h is the h-th element of v.

From the above equations, the definition of tangent cone (eq. 6) and eq. 1,

$$v \in \mathcal{T}(x;\mathbf{H}) \implies \{v^T \Delta a(x) = \sum_{h \in \{1,\ldots,|H|\}} v_h = 0 \quad v^T \Delta b_h(x) = -v_h \leq 0\}$$

Thus from the definition of Linearization cone (eq. 7), $v \in \mathcal{L}(x;\mathbf{H})$.

Now to prove the vice-versa, consider a point $v \in \mathcal{L}(x;\mathbf{H})$. Suppose $v_h < 0$ for some $h \in H$. Let us define $\tau := max\{-x_h/v_h,\ \forall h \in H : v_h < 0\}$. This implies $x + \tau v \succeq 0,\ \forall h \in H$. With this result, we can obtain a sequence of $\tau > \tau^k \downarrow 0$ and $y^k = x + \tau^k v$. From eq. 9 and eq. 1, $\{y^k\} \in \mathbf{H}$. This in turn implies $v \in \mathcal{T}(x;\mathbf{H})$. The result can similarly be checked for the case when $v \succeq 0$, thus completing the proof. □

C. Some relevant definitions

Ukkusuri et al. (2012) provides a self-contained description of some relevant properties regarding set-valued maps. A part of that description is reproduced here for the sake of completion.

A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be

a) closed at a point $\bar{x}$, if $\{x^k\} \to \bar{x} \implies \{y^k(\in F(x^k))\} \to \bar{y}$.
b) closed (convex, compact, contractible) valued at a point $\bar{x}$ if $F(\bar{x})$ is a closed (convex, compact, contractible) set.
c) upper semi-continuous at a point $\bar{x}$ if for every open set $\mathcal{U}$ such that $F(\bar{x}) \cap \mathcal{U} \neq \emptyset$, there exists an open neighborhood $\mathcal{N}$ of $\bar{x}$ such that, for each $x \in \mathcal{N}$, $F(x) \cap \mathcal{U} \neq \emptyset$.
d) bounded on a set C if $\bigcup_{x \in C} F(x)$ is bounded.