GENERALIZATION OF BECKMANN'S TRANSFORMATION FOR TRAFFIC ASSIGNMENT MODELS WITH ASYMMETRIC COST FUNCTIONS

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ABSTRACT

An optimization model is developed to solve the deterministic traffic assignment problem under congested transport networks with cost functions that have an asymmetric Jacobian. The proposed formulation is a generalization of Beckmann's transformation that can incorporate network links with multivariate vector cost functions to capture the asymmetric interactions between the flows and costs of the different links. The objective function is built around a line integral that generalizes the simple definite integral in Beckmann's transformation and is parameterised to ensure the solution of the new problem satisfies Wardrop's first principle of network equilibrium. It is shown that this method is equivalent to the variational inequality approach.

Keywords: Wardrop; traffic equilibrium; asymmetric cost; line integral; integration path; variational inequality; fixed point.

1. INTRODUCTION

This article develops a non-linear optimization model for solving the deterministic traffic assignment problem under congested transport networks that incorporates the general case of network links with asymmetric multivariate cost functions (i.e., the cost vector has an asymmetric Jacobian matrix). The formulation we propose is thus a generalization of the classic Beckmann transformation (Beckmann et al., 1956) in which the objective function is based on a line integral instead of a simple definite integral. The constraints, however, remain exactly the same.

To solve our proposed optimization problem, we devise a parameterisation that ensures the solution satisfies Wardrop's first principle of traffic network equilibrium (Wardrop, 1952). This is proved by a new theorem. We also show that with this parameterisation, the problem's optimality conditions are equivalent to the typical variational inequalities formulated for solving traffic assignment problems with asymmetric interactions. Traditionally, these problems have been addressed either by solving the variational inequality or applying some alternative approach such as fixed-point methods, diagonalisation, decomposition (partitionable, transfer, simplicial, cobweb, etc.), relaxation methods or projection methods.

Notable among works analyzing the formulation of the equilibrium problem, the uniqueness of the solution and the solution algorithm are Dafermos and Sparrow (1969), Smith (1979), Dafermos (1980; 1982), Florian and Spiess (1982), Fisk and Nguyen (1982), Fisk and Boyce (1983), Nagurney (1984), Hammond (1984), Nguyen and Dupuis (1984), Marcotte and Guelatt (1988), Auchmuty (1989), Gabriel and Bernstein (1997) and Patriksson (1998). More recent analyses, focusing for the most part on the design, implementation and comparison of solution algorithms for the traffic assignment problem, including cases with asymmetric interactions, are found in Chen et al. (2002), Panicucci et al. (2007) and Sancho et al. (2015).

In De Grange and Muñoz (2009), the authors proposed the idea of parameterising a line integral in the objective function for a problem similar to Beckmann's transformation in order to tackle the asymmetric multivariate cost function case. In that paper, however, we did not generalize the result, limiting ourselves to particular examples. What was only a research proposal in that work will now be demonstrated here.

The remainder of this article is divided into three sections. Section 2 introduces the formulation of a generalized traffic assignment model with multivariate cost functions having an asymmetric Jacobian as an optimization problem with an objective function containing a line integral. Also explained in this section is the parameterisation (or integration path) for transforming the line integral into a summation of definite integrals from which a direct solution of the problem is obtained that is consistent with Wardrop's first principle. Section 3 presents a numerical example of the proposed model and Section 4 wraps up with a brief summary of our main conclusions.

2. THE MODEL, ITS PARAMETRIZATION AND OPTIMALITY CONDITIONS

Consider the following general formulation of the deterministic traffic assignment problem for a transport network:

$$\min_{\{\mathbf{F}\}} \quad Z = \oint_P C(x) \cdot dx = \sum_{a \in \mathcal{L}} \oint_P c_a(x_1, x_2, ..., x_L) dx_a \tag{1}$$

$$s.t.: \qquad \bar{F} \in \Omega \tag{2}$$

where $\mathcal{L}$ is the set of network links and $L = |\mathcal{L}|$, P is an integration path from flow vector $F_0 := (f_{0,a})_{a \in \ell}$ to flow vector $\bar{F} := (\bar{f}_a)_{a \in \ell}$, Ω is the typical set of flow conservation and non-negativity equations for the links and the corresponding trip matrix, c_a is the cost function of link a dependent on its own flow $\bar{f}_a$ and that of all the other links in the network $\left(c_a = c_a\left(\bar{f}_1, \bar{f}_2, ..., \bar{f}_L\right)\right)$, and $C(x) = (c_1(x), c_2(x), ..., c_L(x))$.

The above expression is clearly a generalization of the objective function in the classic Beckmann transformation, which is defined simply as $\sum_{a \in \mathcal{L}} \int_0^{\bar{f}_a} c_a(x_a) dx_a$, a particular case of (1).

To define the set Ω, we first assume that our network can be represented by a graph $(\mathcal{N}, \mathcal{L})$, where $\mathcal{N} = \{1, 2, ..., N\}$ is the set of nodes and $\mathcal{L}$ the set of links, with $L \geq N$.

We define a set O-D of origin-destination pairs, each pair $r := ij$ of which has a vector E^r such that $E_k^r = 0$ if $k \notin \{i, j\}$, and $E_i^r = -D^r$, $E_j^r = D^r$, where D^r is the flow between pair r. For each pair $r := ij$, the flow in link a generated by trips from origin i to destination j is f_a^r. We thus have the equality $\bar{f}_a = \sum_{r \in O-D} f_a^r$. If we now define $F^r = (f_a^r)_{a \in L}$, we get $\bar{F} = \sum_{r \in O-D} F^r$. Moreover we set $F = (F^r)_{r \in O-D}$.

Finally, we define an incidence matrix $\Delta := (\delta_{na})_{n \in \mathcal{N}, a \in \mathcal{L}}$ in the following manner:

$$\delta_{na} = \begin{cases} 1 & \text{if link } a \text{ enters node } n \\ -1 & \text{if link } a \text{ exits node } n \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

We can now define set Ω as

$$\Omega = \left\{ \overline{F} \in \mathbb{R}^L \middle| \overline{F} = \sum_{r \in O-D} F^r, \Delta F^r = E^r, F^r \geq 0 \right\}. \tag{4}$$

Assume that $\Omega \neq \emptyset$. If the Jacobian matrix of C is symmetric, the integral in (1) can be written as $\oint_P C(x) \cdot dx = V(\overline{F}) - V(F_0)$, where V is a potential function of C, that is, $\nabla V(\overline{F}) = C(\overline{F})$. Then the optimality conditions of problem (1)-(2) can be written directly as follows:

$$\begin{aligned}
&C(\overline{F}) + {}^t\Delta\lambda^r - \mu^r = 0 \\
&\Delta F^r = E^r \\
&\mu^r \otimes F^r = 0 \\
&F^r \geq 0 \\
&\mu^r \geq 0
\end{aligned} \tag{5}$$

If, however, the Jacobian of C is not symmetric, the integral in (1) will depend on the integration path P. In what follows, we propose a way of interpreting the integral and then build an equilibrium model whose every solution satisfies the conditions in (5).

To define the parameterisation of the problem – in effect, the line integral in the objective function (1) – consider a curve $P_0 := \{ x(t) | t \in R \} \subset R^L$ that satisfies the following properties:

i) $x(0) = 0, x'_a(t) > 0$, for all $a \in L$, $t \in R$

ii) $\lim_{t \to +\infty} x_a(t) = +\infty$ and $\lim_{t \to -\infty} x_a(t) = -\infty$

Consider also two flow vectors F_0 and $\overline{F}$. For every link a, there exists a unique number $p_a(f_{0,a}, \overline{f}_a)$ such that $f_{0,a} + x_a(p_a(f_{0,a}, \overline{f}_a)) = \overline{f}_a$. This is so because x_a is a bijective function from R to R. Therefore, F belongs to curve $F_0 + P_0$ if and only if $p_1(f_{0,1}, \overline{f}_1) = \cdots = p_L(f_{0,L}, \overline{f}_L)$. If this holds, its value is denoted $p(F_0, \overline{F})$. Since the derivative of x_a is different from 0 over all R, by the global inverse function theorem, the function p_a is differentiable on R^2.

It follows that if P is an integration path between F_0 and $\overline{F}$ along the curve $F_0 + P_0$, the integral in (1) becomes

$$\begin{aligned}
&\oint_P C(x) \cdot dx = \int_0^{p(F_0, \overline{F})} C(F_0 + x(t)) \cdot x'(t) dt = \sum_{a \in L} \int_0^{p(F_0, \overline{F})} c_a(F_0 + x(t)) x'_a(t) dt \\
&\oint_P C(x) \cdot dx = \sum_{a \in L} \int_0^{p_a(f_{0,a}, \overline{f}_a)} c_a(F_0 + x(t)) x'_a(t) dt
\end{aligned} \tag{6}$$

The optimization problem (1)-(2) can then be written as

$$\min \sum_{a\in L} \int_0^{p_a\left(f_{0,a},\overline{f}_a\right)} c_a\left(F_0 + x(t)\right) x'_a(t)dt$$

$$\begin{aligned} s.t.:\ & \Delta F^r = E^r \ ,\ \forall\ r \in\ O\text{-}D \qquad \left(\lambda^r\right) \\ & F^r \geq 0 \ ,\ \forall\ r \in\ O\text{-}D \qquad \left(\mu^r\right) \\ & \overline{F} = \sum_{r\in O-D} F^r \end{aligned} \tag{7}$$

Note that this formulation does not include a Lagrange multiplier for the constraint $\overline{F} = \sum_{r\in O-D} F^r$ given that this equality can be included directly in the objective function.

The optimality conditions for (7) are

$$\begin{aligned} & \frac{\partial p_a\left(f_{0,a},\overline{f}_a\right)}{\partial \overline{f}_a} \cdot c_a\left(F_0 + x\left(p_a\left(f_{0,a},\overline{f}_a\right)\right)\right) \cdot x'_a\left(p_a\left(f_{0,a},\overline{f}_a\right)\right) + \left({}^t\Delta\lambda^r\right)_a - \mu^r{}_a = 0, \forall\left(a\in L, r\in O-D\right) \\ & \Delta F^r = E^r, \forall r \in O-D \\ & \mu^r \otimes F^r = 0, \forall r \in O-D \\ & F^r \geq 0, \forall r \in O-D \\ & \mu^r \geq 0, \forall r \in O-D \end{aligned}$$

where $\otimes$ is the Hadamard product, which for two vectors v and w of the same dimension gives the product $v \otimes w = \left(v_1 w_1, v_2 w_2, ..., v_m w_m\right)$, and $\left({}^t\Delta\lambda^r\right)_a$ is component a of vector ${}^t\Delta\lambda^r$.

If we calculate $\frac{\partial p_a\left(f_{0,a},\overline{f}_a\right)}{\partial \overline{f}_a}$ we obtain $f_{0,a} + x_a\left(p_a\left(f_{0,a},\overline{f}_a\right)\right) = \overline{f}_a$, implying that differentiating with respect to $\overline{f}_a$ gives $x'_a\left(p_a\left(f_{0,a},\overline{f}_a\right)\right)\frac{\partial p_a\left(f_{0,a},\overline{f}_a\right)}{\partial \overline{f}_a} = 1$. The optimality conditions above then become

$$\begin{aligned} & c_a\left(F_0 + x\left(p_a\left(f_{0,a},\overline{f}_a\right)\right)\right) + \left({}^t\Delta\lambda^r\right)_a - \mu^r{}_a = 0, \forall a \in L, \forall r \in O-D \\ & \Delta F^r = E^r, \forall r \in O-D \\ & \mu^r \otimes F^r = 0, \forall r \in O-D \\ & F^r \geq 0, \forall r \in O-D \\ & \mu^r \geq 0, \forall r \in O-D \end{aligned} \tag{8}$$

It can be seen that the set of solutions for (8) depends on the flow vector F_0 and that in general, they do not belong to the curve $F_0 + P_0$. If there exists a solution F that does belong to the curve, then for all links a it is the case that $F_0 + x\left(p_a\left(f_{0,a}, \bar{f}_a\right)\right) = F_0 + x\left(p\left(F_0, \bar{F}\right)\right) = \bar{F}$. Also observable is that $\bar{F}$ belongs to $F_0 + P_0$ if and only if $F_0 + x\left(p_1\left(f_{0,1}, \bar{f}_1\right)\right) = \bar{F}$.

Thus, (8) can be written as the following system of equations with unknowns F and F_0:

$$
\begin{aligned}
&C(\bar{F}) + {}^t\Delta\lambda^r - \mu^r = 0, \forall r \in O-D \\
&\Delta F^r = E^r, \forall r \in O-D \\
&\mu^r \otimes F^r = 0, \forall r \in O-D \\
&F^r \geq 0, \forall r \in O-D \\
&\mu^r \geq 0, \forall r \in O-D \\
&F_0 + x(p_1(f_{0,1}, \bar{f}_1)) = \bar{F}
\end{aligned} \tag{9}
$$

The following proposition allows us to eliminate the unknown F_0:

- **Proposition**: A flow vector F is a solution of the system

$$
\begin{aligned}
&C\left(\bar{F}\right) + {}^t\Delta\lambda^r - \mu^r = 0 \\
&\Delta F^r = E^r \\
&\mu^r \otimes F^r = 0 \\
&F^r \geq 0 \\
&\mu^r \geq 0
\end{aligned} \tag{10}
$$

  if and only if there exists a flow vector F_0 such that $\left(F, F_0\right)$ is a solution of system (9).

- **Proof**: If $\left(F, F_0\right)$ is a solution of system (9) then F is a solution of system (10). If F is a solution of system (10) then $\left(F, \bar{F}\right)$ is a solution of system (9).

The following proposition relates system (10) to a variational inequality.

- **Proposition**: A flow vector F is a solution of system (10) if and only if $\bar{F} \in \Omega$ and for all $\bar{F}' \in \Omega$, $C\left(\bar{F}\right) \cdot \left(\bar{F}' - \bar{F}\right) \geq 0$.

- **Proof:** F is a solution of system (10) if and only if $\bar{F}$ is a solution of the following optimization problem:

$$\min_{\bar{F}' \in \Omega} C\left(\bar{F}\right)\cdot \bar{F}'$$

This is equivalent to stating that $\bar{F}$ is the solution of the following variational inequality:

$$\forall \bar{F}' \in \Omega \,,\; C\left(\bar{F}\right)\cdot\left(\bar{F}' - \bar{F}\right) \geq 0$$ □

Finally, the theorem below proves that the system of optimality conditions (9) satisfies Wardrop's first principle. Before developing the proof we set out the necessary notation. For each O-D pair $r := ij$, a route p between i and j consists of a finite sequence of links $\left(a_0, a_1, ..., a_q\right)$ such that i is the origin of link a_0, j is the destination of link a_q and for all $k \in \{0,1,...,q\}$, the destination of link a_k is the origin of link a_{k+1}. The quantity h_p^r represents the flow on route p between origin i and destination j and is defined as $h_p^r = \min_{k=0,...,q} f_{a_k}^r$. The cost of route p between O-D pair r is given by $g_p^r\left(\bar{F}\right) = \sum_{k=0}^{q} c_{a_k}\left(\bar{F}\right)$.

- **Theorem**: Let F be a solution of (10) and $r := ij$ an origin-destination pair. Also let p and p' be two routes between origin i and destination j. If $h_p^r \neq 0$ and $h_{p'}^r \neq 0$, then $g_p^r\left(\bar{F}\right) = g_{p'}^r\left(\bar{F}\right)$.

- **Proof**: Let $\lambda^r := \left(\lambda_n^r\right)_{n \in \mathcal{N}}$ be a vector of Lagrange multipliers associated with the subset of constraints $\Delta F^r = E^r$. Recall that route p can be written as $p = \left(a_0, a_1, ..., a_q\right)$. For all $k \in \{0,...,q\}$, we have

$$c_{a_k}\left(\bar{F}\right) + \left({}^t\Delta\lambda^r\right)_{a_k} - \mu_{a_k}^r = 0 \tag{11}$$

If $h_p^r \neq 0$, then $f_{a_k}^r > 0$ for all $k \in \{0,1,...,q\}$, implying in turn that $\mu_{a_k}^r = 0$. Also, $\left({}^t\Delta\lambda^r\right)_{a_k} = \sum_{n \in \mathcal{N}} \delta_{n a_k} \lambda_n^r$.

Now let there be nodes $\left(n_0, n_1, ..., n_{q+1}\right)$ such that for each $k \in \{0,1,...,q\}$, link a_k exits node n_k and enters node n_{k+1}. Then for all $k \in \{0,1,...,q\}$, we have

$$\sum_{n \in \mathcal{N}} \delta_{n a_k} \lambda_n^r = \lambda_{n_{k+1}}^r - \lambda_{n_k}^r \tag{12}$$

Therefore equality (11) it follows that

$$c_{a_k}\left(\overline{F}\right) = \lambda^r_{n_k} - \lambda^r_{n_{k+1}} \tag{13}$$

we then obtain

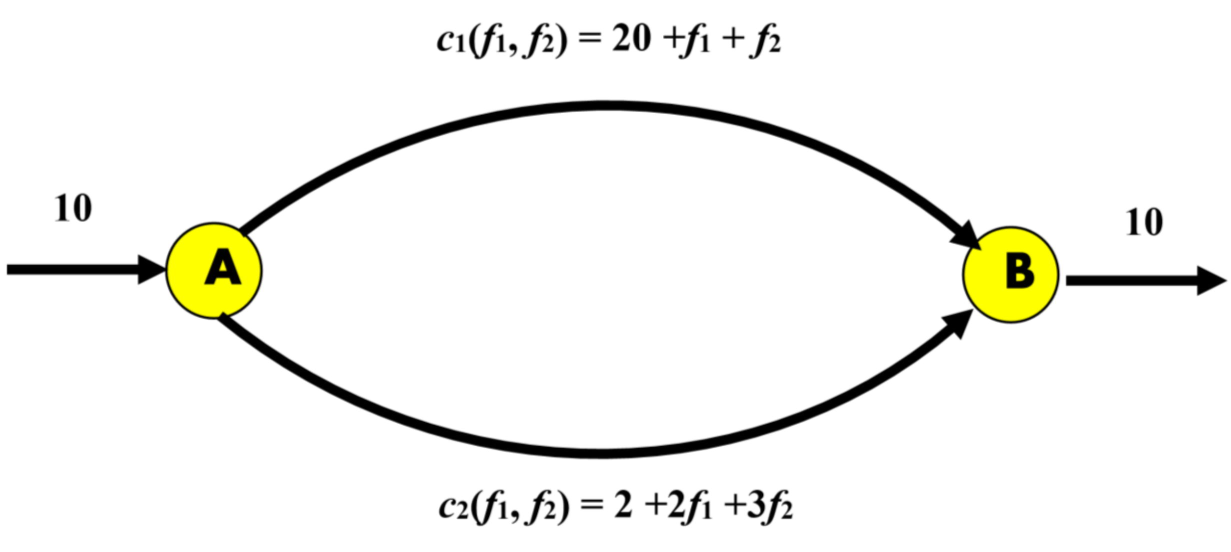
$$g^r_p(\overline{F}) = \sum_{k=0}^{q} c_{a_k}\left(\overline{F}\right) = \lambda^r_{n_0} - \lambda^r_{n_{q+1}} = \lambda^r_i - \lambda^r_j \tag{14}$$

Since the Lagrange multiplier vector λ^r does not depend on the route between origin *i* and destination *j*, we deduce that sum (14) above is also independent of that route.

Observe also that the Lagrange multiplier set is not unique but the quantities $\lambda^r_i - \lambda^r_j$ do not depend on the choice of Lagrange multipliers. These quantities are interpreted as the trip cost between origin *i* and destination *j*.

Example: Assume there is a network with two nodes A and B and two parallel links 1 and 2 that join them (see Figure 1). The links' respective asymmetric cost functions are $c_1\left(f_1, f_2\right) = 20 + f_1 + f_2$, $c_2\left(f_1, f_2\right) = 2 + 2f_1 + 3f_2$. Assume also that total demand between the two nodes is 10 and is distributed between the two links.

Figure 1
Example Network with Asymmetric Cost Functions



Given the cost functions as just defined, the equilibrium flows f_1 and f_2 that satisfy them, or in other words, that satisfy Wardrop's first principle, are $\left(f_1, f_2\right) = (2,8)$.

This equilibrium is determined using our construction with the line integral in the following manner. Let $x(t)=(at,bt)$ be a vector function where a > 0 and b > 0 are real numbers. If $F_0=(f_{0,1},f_{0,2})$ is a flow vector, problem (7) can then be written as

$$\min_{\substack{f_1,f_2\geq 0\\ f_1+f_2=10}} \quad a\int_0^{\frac{f_1-f_{0,1}}{a}}\left(20+f_{0,1}+f_{0,2}+(a+b)t\right)dt+b\int_0^{\frac{f_2-f_{0,2}}{b}}\left(2+2f_{0,1}+3f_{0,2}+(2a+3b)t\right)dt \qquad (15)$$

As we saw above, solving the traffic assignment problem is the equivalent of finding a flow vector F_0 such that $F = F_0$, where F is a solution of (15). Assume then that $F_0 \in \mathbf{\Omega}$, which implies that $f_{0,1}+f_{0,2}=10$. Eliminate variable f_2 by replacing it with $f_2 = 10 - f_1$ and simplify using $X = f_1 - f_{0,1}$. Problem (15) then becomes

$$\min_{-f_{0,1}\leq X\leq 10-f_{0,1}} \quad a\int_0^{\frac{X}{a}}\left(30+(a+b)t\right)dt+b\int_0^{-\frac{X}{b}}\left(32-f_{0,1}+(2a+3b)t\right)dt \qquad (16)$$

The upper bound of the second integral in (16) is a consequence of the following equalities:

$$f_2-f_{0,2}=10-f_1-(10-f_{0,1})=f_{0,1}-f_1=-X$$

The objective function of (16) is a convex quadratic function and its unique solution is given by

$$X=\min\left(\frac{ab}{2a^2+4ab+b^2}\left(2-f_{0,1}\right),10-f_{0,1}\right)$$

This implies that

$$f_1=\min\left(\frac{ab}{2a^2+4ab+b^2}\left(2-f_{0,1}\right)+f_{0,1},10\right)$$

The function $f_{0,1}\to f_1$ where f_1 is given by the equality immediately above is a Lipschitz function with a Lipschitz constant equal to $\frac{2a^2+3ab+b^2}{2a^2+4ab+b^2}$. Since the latter is less than 1, the function is a contraction and therefore has a unique fixed point. Every sequence defined by

$$f_1^n=\min\left(\frac{ab}{2a^2+4ab+b^2}\left(2-f_1^{n-1}\right)+f_1^{n-1},10\right)$$

converges to this point, which is equal to $f_1 = 2$. We then obtain the value for $f_2 = 10 - f_1 = 8$.

This little example gives a simple demonstration of the operation of our generalization of Beckmann's transformation based on the line integral defined in (1), which allows us to obtain a fixed-point iteration that converges.

3. ANALYSIS OF A NUMERICAL EXAMPLE

We now solve the system of equations in (10) for a numerical example. We begin by defining a function $H^r : R^{2L+N} \to R^{2L+N}$ as

$$H^r\left(F, \lambda^r, \mu^r\right) = \begin{pmatrix} C\left(\bar{F}\right) + {}^t\Delta\lambda^r - \mu^r \\ \Delta F^r - E^r \\ F^r \otimes \mu^r \end{pmatrix} \tag{17}$$

Solving (10) is equivalent of finding (F, λ, μ) such that $H^r\left(F, \lambda^r, \mu^r\right) = 0$ for all O-D pairs r and $F^r \geq 0$, $\mu^r \geq 0$ Since $\ker\left({}^t\Delta\right) = R(1,...,1)$, the set of multipliers λ^r will not be unique unless we set the value of λ_n^r for a certain node n. We therefore define the function $H_n^r : R^{2L+N} \to R^{2L+N+1}$ as

$$H_n^r\left(F, \lambda^r, \mu^r\right) = \begin{pmatrix} C\left(\bar{F}\right) + {}^t\Delta\lambda^r - \mu^r \\ \Delta F^r - E^r \\ F^r \otimes \mu^r \\ \lambda_n^r \end{pmatrix} \tag{18}$$

and H_n as $H_n\left(F, \lambda, \mu\right) := \left(H_n^r\left(F, \lambda^r, \mu^r\right)\right)_{r \in O-D}$.

For greater clarity we use the notation $x = (F, \lambda, \mu) \in R^{R(2L+N)}$ and $K = \left\{ x = (F, \lambda, \mu) \in R^{R(2L+N)} \middle| f_a^r > 0, \mu_a^r > 0, \forall a \in L, \forall r \in O-D \right\}$, where R is the cardinal of O-D.

To solve the equation $H_n(x) = 0, x \in \bar{K}$, we apply a potential reduction method (for a detailed treatment, see Wang et al., 1996). The cost functions for our example take the following form:

$$c_a\left(f_1, ..., f_L\right) = e_a + \sum_{a'=1}^{L} b_{a,a'} f_{a'} + g\left(\alpha_a, \beta_a, f_a^{\lim}, f_a\right) \tag{19}$$

where $e_a, b_{a,a'}, \alpha_a, \beta_a, f_a^{\lim}$ are positive constants and the function *g* is defined by

$$g\left(\alpha, \beta, f^{\lim}, f\right) = \alpha \ln\left(1 + \exp\left(\frac{\beta\left(f - f^{\lim}\right)}{\alpha}\right)\right) \tag{20}$$

Also, *g* satisfies the following property:

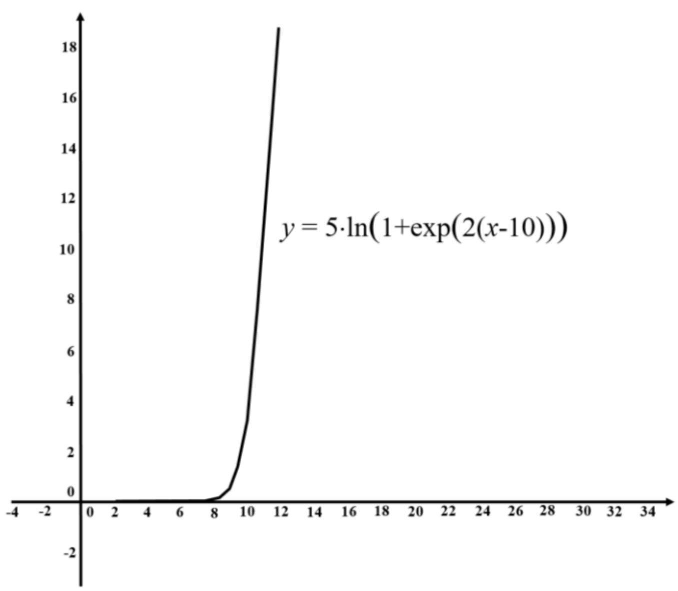
$$\lim_{\alpha \to 0^+} g\left(\alpha, \beta, f^{\lim}, f\right) = \max\left(0, \beta\left(f - f^{\lim}\right)\right) \tag{21}$$

Therefore, the cost function for link *a* satisfies

$$c_a\left(f_1, \ldots, f_L\right) \approx e_a + \sum_{a'=1}^{L} b_{a,a'} f_{a'} + \max\left(0, \beta_a\left(f_a - f_a^{\lim}\right)\right) \tag{22}$$

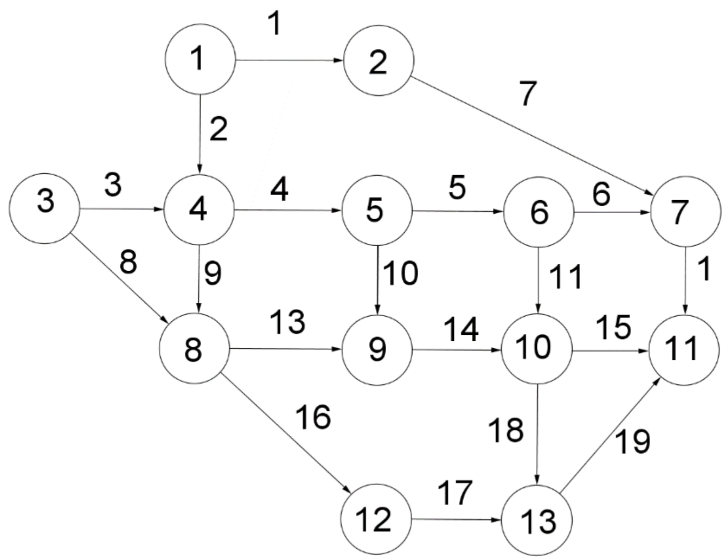
where $f_a^{\lim}$ is the threshold flow beyond which congestion begins to appear and *g* is the cost that congestion generates. In the single variable case, a cost function of the type in (20) has the shape graphed in Figure 2.

Figure 2
Cost Function (Univariate Case)



The network used for our numerical example was inspired by the classic version due to Nguyen and Dupuis (1984). We assume that there are four O-D pairs (r), each with its respective fixed demand level (D), as follows: $r_1 = 1-13, D^{r_1} = 40$; $r_2 = 1-11, D^{r_2} = 70$; $r_3 = 3-13, D^{r_3} = 30$; $r_4 = 3-11, D^{r_4} = 40$. A visual representation of the network with its various links and nodes is given in Figure 3.

Figure 3
Network for the Numerical Example: 13 Nodes and 19 Links



The (asymmetric) cost functions for the links are set out in Table 1.

Table 1
Cost Functions for the Example Network

$c_1 = 1 + 2f_2 + g(5,10,30, f_1)$	$c_{11} = 1 + f_{11} + 3f_{14} + g(5,10,10, f_{11})$
$c_2 = 1 + 4f_3 + f_2 + g(5,10,20, f_2)$	$c_{12} = 1 + f_{12} + g(5,10,40, f_{12})$
$c_3 = 1 + f_3 + g(5,10,20, f_3)$	$c_{13} = 1 + f_{13} + g(5,10,20, f_{13})$
$c_4 = 1 + f_4 + g(5,10,20, f_4)$	$c_{14} = 1 + f_{14} + g(5,10,20, f_{14})$
$c_5 = 1 + f_5 + g(5,10, 20, f_5)$	$c_{15} = 1 + f_{15} + g(5,10,20, f_{15})$
$c_6 = 1 + f_6 + 4f_7 + g(5,10,20, f_6)$	$c_{16} = 2 + f_{16} + g(5,10,40, f_{16})$
$c_7 = 8 + 2f_7 + g(5,10,40, f_7)$	$c_{17} = 1 + f_{17} + g(5,10,40, f_{17}) + 8f_{18}$
$c_8 = 10 + f_8 + g(5,10,40, f_8)$	$c_{18} = 1 + 0.5f_{18} + g(5,10,10, f_{18})$
$c_9 = 1 + 3f_8 + f_9 + g(5,10,40, f_9)$	$c_{19} = 1 + f_{19} + g(5,10,40, f_{19})$
$c_{10} = 1 + f_{10} + 4f_{13} + g(5,10,10, f_{10})$	

The convergence criterion is $\|H(x)\| < 10^{-5}$. The parameter value chosen for α_a is relatively large so that the second derivative of the costs functions will not be too high. The plot in Figure 2 is the graph obtained for the function $g(5,10,10, f)$.

The equilibrium flows and costs for the 19 links in our example are set out in Table 2 while the equilibrium costs for the routes used between each O-D pair are given in Table 3

Table 2
Equilibrium Flows and Costs for the Example Network Links

Link	$f_a^{r_1}$	$f_a^{r_2}$	$f_a^{r_3}$	$f_a^{r_4}$	$\bar{f}_a = \sum_{i=1}^{4} f_a^{r_i}$	$C_a\left(\bar{F}\right)$
1	0	63.6334	0	0	63.6334	464.6011
2	40	6.3666	0	0	46.3666	373.6884
3	0	0	5.5303	10.1337	15.664	16.6649
4	17.6199	3.9334	4.3787	9.0756	35.0076	186.0841
5	12.8313	2.4145	2.8562	6.9220	25.024	76.2649
6	0	0	0	0	0	255.5337
7	0	63.6334	0	0	63.6334	371.6011
8	0	0	24.4697	29.8663	54.336	207.6958
9	22.3801	2.4332	1.1516	1.0581	27.023	191.0309
10	4.7886	1.5189	1.5225	2.1536	9.9836	133.4656
11	12.8313	2.4145	2.8562	6.9220	25.024	295.5388
12	0	63.6334	0	0	63.6334	300.9677
13	1.5838	1.0935	2.3590	24.7381	29.7744	128.5188
14	6.3724	2.6123	3.8815	26.8917	39.7579	238.3381
15	0	3.9673	0	32.8139	36.7812	205.5939
16	20.7963	1.3397	23.2622	6.1862	51.5844	169.4295
17	20.7963	1.3397	23.2622	6.1862	51.5844	392.4360
18	19.2037	1.0595	6.7378	0.9998	28.0008	195.0086
19	0	2.3992	0	7.1861	9.5853	10.5853

Table 3
Equilibrium Costs for the Routes Used between each Network Example O-D Pair

Origin-Destination	Route	Route cost
1-13	2-4-5-11-18	1126.6
1-13	2-9-13-14-18	1126.6
1-13	2-9-16-17	1126.6
1-11	1-7-12	1137.2
1-11	2-4-5-11-15	1137.2
1-11	2-9-16-17-19	1137.2
3-13	3-4-5-11-18	769.56
3-13	3-9-13-14-18	769.56
3-13	8-16-17	769.56
3-11	3-4-5-11-15	780.15
3-11	3-9-13-14-15	780.15
3-11	8-16-17-19	780.15

It can be easily verified that these equilibrium conditions satisfy Wardrop’s first principle.

4. CONCLUSIONS

This article presented a generalization of the classic Beckmann transformation for specifying and solving the traffic assignment problem. The proposed optimization problem is similar to Beckmann's except that the objective function is built around a line integral that incorporates a cost vector function with multivariate and asymmetric costs. The constraints in the two versions are identical. To solve the optimization problem the line integral must be parameterised, which is done by defining an integration path based on a flow vector. This converts the new formulation into a fixed-point equilibrium problem. Its solutions are also the solutions of a classic variational inequality and satisfy Wardrop's first principle.

The proposed approach contributes to a better understanding of equilibrium in transport networks and provides a common traffic assignment framework for analyzing both the simple case captured by Beckmann's transformation and the more general case represented by a variational inequality. The theoretical development of the new optimization problem is complemented by a numerical example based on the classic traffic network due to Nguyen and Dupuis but with multivariate cost functions for the network links and asymmetric interactions. It was shown that the solution of the problem via a parameterised line integral generates a fixed-point equilibrium condition equivalent to a variational inequality whose solution satisfies Wardrop's first principle.

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