Land use planning and optimal subsidies *

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Abstract

The urban plannification is a complex problem involving the choice of a social objective for a built city, from which an optimal allocation of agents should be found as a market equilibrium with the help of some policy like subsidies. In order to find the optimal allocation and subsidies, we prove a fundamental result which asserts that any feasible allocation can be achieved as a market equilibrium by applying suitable subsidies, which can be computed even in the case with location externalities. This allow us to split the problem in two independent steps. First, we find the optimal allocation for a social objective and, second, we use the fundamental result to derive subsidies that reproduce the optimal allocation as a market equilibrium. The computation of the optimal allocation is obtained from a convex optimization urban planning problem applicable to a wide class of objective functions. Optimal subsidies can be obtained even when the policy maker seeks to reduce the impact of the policy, or faces implementation constraints, e.g., some agents or zones can not be subsidized or taxed, or in some specific zones the budget for subsidies is limited. As an example, we simulate a small city which aims at improving social inclusion.

Keywords: convex optimization; land use planning problem; location subsidies; urban segregation.

1. INTRODUCTION

Megacities face chronic problems like congestion, social segregation, urban sprawl, and high land rents, in addition to crime and the concern about climate change. These can be seen as costs of development, but there are also opportunities for the decision maker to implement policies that reduces this negative impacts. Methods to study how to plan cities have so far concentrated on simulating the long term impacts of project and policies defined as future scenarios. Land use and transport models contribute in this task forecasting the impact of each scenario considered and is fair to say that these models have advanced in the last two decades to become operational and widely used by practitioners, which can be seen in Hunt et al. (2005); Preston et al. (2010); Timmermans and Zhang (2009); and Wegener (1994, 1998).

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However, the scenario approach leaves the planner with the enormous task of building wise scenarios. This is a complex problem because potential subsidies and projects in the urban context can be numerous and testing each scenario is computationally costly. Consider for example a planner who has to define appropriate subsidies in a city with M socioeconomic groups and N zones. To achieve some specific social goal, she has to define $N \times M$ subsidies/taxes. For example, in a small city with $M = 10$ and $N = 1.000$ we have 1×10^4 location subsidies. In sum, the problem of setting optimal subsidies is not really feasible; there is a need to develop models able to optimize the city by efficiently searching in the large number of scenarios.

The more simple task of modeling integrated land use and transport (LUT) scenarios is already a complex mathematical problem, which researchers have found difficult to solve providing confidence on the results in the presence of location externalities and transport congestion. Some models, e.g. Ma and Lo (2012), solve the equilibrium of the integrated LUT problem, including congestion and without location externalities, but they cannot assure uniqueness of the solution. Briceño-Arias et al. (2008) solved the LUT problem for a unique solution, extended by Bravo et al. (2010) for the case with location externalities. However, none of these models attempt to optimize the city location distribution regarding a social goal. This might be justified by the extra difficulty of adding to the LUT problem the complexity of finding an optimal allocation and the corresponding subsidies.

The above difficulties inspire us to simplify in this paper the planning and subsidies problem by assuming transportation costs as given. In this context, our approach is to decouple the planning problem from the calculation of optimal subsidies reducing the complexity of each step. Therefore, in the first step we find an optimal allocation $\boldsymbol{x}^*$, which is the solution to the land use planning problem; this problem may be defined by a large class of objective functions including, for instance, social inclusion measures. Second, we compute the optimal subsidies by zone and agent, using a fundamental result, which asserts that any feasible allocation can be achieved by some set of subsidies. That is, by applying this set of subsidies, agents behaves freely in the location market and the resulting equilibrium is $\boldsymbol{x}^*$.

In Section 2, we formulate the planning problem in terms of its primal and dual problems, where the land-use equilibrium problem is a particular case. We also prove existence and uniqueness of this optimal primal-dual solution under suitable conditions. In Section 3, we compute subsidies which make the optimum allocation $\boldsymbol{x}^*$ to be an equilibrium and we propose several policies to define these subsidies. In Section 4 we study an application to the social inclusion problem, we consider cases where the policy maker faces some implementation constraints and we provide numerical simulations.

2. THE PLANNING PROBLEM

In this section we formulate the planning problem as a primal-dual formulation for a wide range of objective functions; the market equilibrium problem is represented by one of them, making this problem a special case of the planning problem. Let C be the set of types of households and suppose that one firm controls the real estate supply. For every zone $i \in N$, let $S_i \in]0, +\infty[$ be the supply in the zone i and, for every $h \in C$, let $H_h \in]0, +\infty[$ be the demand of the households type h in the land use market. For every $h \in C$ and $i \in N$, we denote by x_{hi} the number of households type h localized in i and we set $z_{hi} \in \mathrm{R}$ be the utility perceived by the household h on the amenities at zone i, including real estate attributes and accessibility to the household activities.

These utilities are assumed to be constant and known. Hence, transportation costs and other location attibutes are assumed to be exogenous, except for location externalities which are studied at the end of Section 3. Additionally we assume the market clearing condition $T = \sum_{i\in N} S_i = \sum_{h\in C} H_h$, i.e., we suppose that the number of households demanding for a zone coincides with the number of houses available.

Let us denote as $\mathscr{C}$ the class of functions $\psi\colon \mathrm{R}\times[0,+\infty[\,\to\,]-\infty,+\infty]$ such that

$$(\forall z\in \mathrm{R})\qquad \begin{cases}\psi(z,\cdot) \text{ is strictly convex and}\\ \lim_{x\to+\infty}\psi(z,x)=+\infty.\end{cases} \tag{2.1}$$

This class defines the set of objective functions that we consider for solving the land use planning problem defined as follows.

Problem 2.1 (Land use planning - primal problem)**.** *For every $h\in C$ and $i\in N$, let $z_{hi}\in \mathrm{R}$, let $\psi_{hi}\in\mathscr{C}$ such that* $\operatorname{dom}\psi_{hi}(z_{hi},\cdot)=[0,a_{hi}[$, *for some $a_{hi}\in\,]0,+\infty]$, let*

$$\Xi=\Big\{\boldsymbol{x}\in\mathrm{R}^{|C|\times|N|}\;\Big|\;(\forall i\in N)\quad \sum_{h\in C}x_{hi}=S_i\quad\textit{and}\quad(\forall h\in C)\quad\sum_{i\in N}x_{hi}=H_h\Big\}, \tag{2.2}$$

and suppose that

$$\Xi\cap\bigtimes_{h\in C}\bigtimes_{i\in N}[0,a_{hi}[\,\neq\varnothing. \tag{2.3}$$

The problem is to

$$\underset{\boldsymbol{x}\in\Xi}{\text{minimize}}\;\sum_{h\in C}\sum_{i\in N}\psi_{hi}(z_{hi},x_{hi}). \tag{2.4}$$

The objective function of this problem belongs to the class $\mathscr{C}$, that is, it is strictly convex and coercive. Additionally we allow functions $(\psi_{hi})_{h\in C,\,i\in N}$ to have as domain all the positive real numbers ($a_{hi}=+\infty$) or, if necessary, have a restricted domain ($a_{hi}<+\infty$) if the function is unbounded.

Proposition 2.2 (Dual problem)**.** *Under the assumptions of Problem 2.1, the dual problem associated to* (2.4) *is*

$$\underset{(\boldsymbol{b},\boldsymbol{r})\in\mathrm{R}^{|C|+|N|}}{\text{minimize}}\;\boldsymbol{\Phi}(\boldsymbol{b},\boldsymbol{r}):=\sum_{h\in C}H_hb_h+\sum_{i\in N}S_ir_i+\sum_{h\in C}\sum_{i\in N}\varphi_{hi}(z_{hi},-b_h-r_i), \tag{2.5}$$

where, for every $h\in C$ and $i\in N$,

$$\varphi_{hi}\colon(z_{hi},u)\mapsto\psi_{hi}(z_{hi},\cdot)^*(u):=\sup_{x\in[0,a_{hi}[}\big(\langle x\mid u\rangle-\psi_{hi}(z_{hi},x)\big). \tag{2.6}$$

In the dual problem (2.5), decision variables $(b_h)_{h\in C}$ and $(r_i)_{i\in N}$ correspond to the Lagrange multipliers associated to the constraints in (2.2) and represent rents and utilities by zone and household type, respectively. On the other hand, the function φ_{hi} defined in (2.6) is known as the Fenchel conjugate of $\psi_{hi}(z_{hi},\cdot)$, which is denoted by $\psi_{hi}(z_{hi},\cdot)^*$ and is also convex. The following example illustrates a particular instance when the Fenchel conjugate is easily computable.

Example 2.3 (Land use market equilibrium)**.** *Let $\mu \in]0,+\infty[$. In the particular case when, for every $h \in C$ and $i \in N$, $\psi_{hi}\colon (z,x) \mapsto -xz + x(\ln x - 1)/\mu \in \mathscr{C}$, Problem 2.1 becomes*

$$\underset{\boldsymbol{x}\in\Xi}{\text{minimize}} \quad \sum_{h\in C}\sum_{i\in N} -x_{hi}z_{hi} + \frac{1}{\mu}x_{hi}(\ln x_{hi} - 1), \tag{2.7}$$

which is the bid-rent equilibrium problem presented in Briceño-Arias et al. (2008), the well known entropy maximizing problem. Hence, this equilibrium problem can be seen as a particular case of our framework. Moreover, in this case it is easy to verify that, for every $(h,i) \in C \times N$, $\varphi_{hi}(z,u) = \psi_{hi}(z_{hi},\cdot)^*(u) = e^{\mu(z+u)}/\mu$. and, hence, (2.5) *becomes*

$$\underset{(\boldsymbol{b},\boldsymbol{r})\in \mathrm{R}^{|C|+|N|}}{\text{minimize}} \quad \sum_{h\in C} H_h b_h + \sum_{i\in N} S_i r_i + \frac{1}{\mu}\sum_{h\in C}\sum_{i\in N} e^{\mu(z_{hi}-b_h-r_i)}, \tag{2.8}$$

which is the dual problem associated to the well known entropy maximization problem (2.7) *(see Briceño-Arias et al. 2008). The first order optimality conditions of* (2.8) *are*

$$(\forall h \in C)(\forall i \in N) \qquad \begin{cases} \sum_{i\in N} e^{\mu(z_{hi}-b_h-r_i)} = H_h \\ \sum_{h\in C} e^{\mu(z_{hi}-b_h-r_i)} = S_i \end{cases} \tag{2.9}$$

and we deduce that the solution to the primal problem is $x_{hi} = e^{\mu(z_{hi}-b_h-r_i)}$.

Note that, for every $(\boldsymbol{b},\boldsymbol{r}) \in \mathrm{R}^{|C|+|N|}$ and $\alpha \in \mathrm{R}$, $\boldsymbol{\Phi}(\boldsymbol{b}+\alpha, \boldsymbol{r}-\alpha) = \boldsymbol{\Phi}(\boldsymbol{b},\boldsymbol{r})$, where $b + \alpha = (b_1+\alpha,\ldots,b_{|C|}+\alpha)$ and $r - \alpha = (r_1-\alpha,\ldots,r_{|N|}-\alpha)$, which follows from the market clearing assumption $\sum_{h\in C} H_h = \sum_{i\in N} S_i$. Hence, for having uniqueness of the solution, we have to consider some additional constraints in the dual problem.

Proposition 2.4 (Uniqueness of the dual solution)**.** *Problem 2.1 has a unique solution. In addition, if the dual problem considers one of the following constraints D_1 to D_4 for some $\eta \in \mathrm{R}$, then the dual problem* (2.5) *has also a unique solution.*

(i) $\boldsymbol{b} \in D_1 = \{\boldsymbol{b} \in \mathrm{R}^{|C|} \mid b_1 = \eta\}$.
(ii) $\boldsymbol{b} \in D_2 = \{\boldsymbol{b} \in \mathrm{R}^{|C|} \mid \frac{1}{|C|}\sum_{h\in C} b_h = \eta\}$.
(iii) $\boldsymbol{r} \in D_3 = \{\boldsymbol{r} \in \mathrm{R}^{|N|} \mid r_1 = \eta\}$.
(iv) $\boldsymbol{r} \in D_4 = \{\boldsymbol{r} \in \mathrm{R}^{|N|} \mid \frac{1}{|N|}\sum_{i\in N} r_i = \eta\}$.

Moreover, primal and dual solutions are related via $\overline{x}_{hi} = \big(\varphi_{hi}(z_{hi},\cdot)\big)'(\overline{b}_h + \overline{r}_i)$, for every $h \in C$ and $i \in N$, where $(\varphi_{hi}(z_{hi},\cdot))_{h\in C,i\in N}$ are defined in (2.6)*. In the rest of this paper we asume $b_1 = 0$ in order to guarantee uniqueness of the primal-dual solution.*

Remark 2.5 (Uniqueness in land use market equilibrium)**.** *We deduce from Proposition 2.4 and Example 2.3 that* (2.7) *and* (2.8) *have unique solutions depending on the utilities $\boldsymbol{z} = (z_{hi})_{h\in C,\, i\in N}$ under an additional condition, let us say, $b_1 = 0$. We will denote by $(x_{hi}(\boldsymbol{z}))_{h\in C,\, i\in N}$ and by $(b_h(\boldsymbol{z}))_{h\in C}$ and $(r_i(\boldsymbol{z}))_{i\in N}$ such solution.*

Remark 2.6 (The random bidding and choice equivalent model)**.** *In the primal and dual problems* (2.7) *and* (2.8)*, $(\boldsymbol{b},\boldsymbol{r}) \in \mathrm{R}^{|C|+|N|}$ represent the Lagrange multipliers of the constraints* (2.2)*, for total households and total supply, respectively. These multipliers can be interpreted considering the equivalent model derived from the random* *bidding framework by Ellickson (1981) and Martínez (1992), where* $\boldsymbol{b}$ *and* $\boldsymbol{r}$ *represent the vectors of utilities and rents that clears the market. Indeed,* (2.9) *can be solved for* r_i *in the second equation and replaced into the solution to obtain the logit model* $x_{hi} = S_i e^{\mu(z_{hi}-b_h)} / \sum_{g\in C} e^{\mu(z_{gi}-b_g)}$*. This solution represents the expected number of agents of type h located at zone i with* S_i *locations;* r_i *is the expected rent obtained in the auction of i;* $z_{hi} - b_h$ *is the expected value of agent h's bid for zone i, with* b_h *the utility level that the agent obtains at zone i at equilibrium. Similarly,* b_h *can be obtained from the first equation in* (2.9) *and by replacing this expression into the primal solution we obtain the logit model involving the household h's surplus, which is the choice model proposed in Anas (1982). This shows the equivalence between the bidding and choice models under conditions* (2.9)*. Note that, from* (2.9) *we can write* $\boldsymbol{r} = f(\boldsymbol{b})$ *and* $\boldsymbol{b} = g(\boldsymbol{r})$*, and combining both equations we derive the fixed point* $\boldsymbol{b} = (g\circ f)(\boldsymbol{b})$ *yielding the equilibrium value of* $\boldsymbol{b}$*, i.e. the utility attained by agents at equilibrium (Martínez and Henríquez 2007). Additionally, the equivalence between the maximum likelihood parameters of the logit model and the double constrained entropy model presented in Example 2.3 was proved by Anas (1983).*

The variety of objectives functions comprised in the planning problem of (2.5) can be solved using the general solution algorithm derived from Briceño-Arias and Combettes (2011).

3. OPTIMAL SUBSIDIES

In this section, we formulate the problem of finding the optimal subsidies $\boldsymbol{s} = (s_{hi})_{h\in C,\, i\in N}$ that should be given to each type of household in each zone such that the agents behaviour is altered in order to attain a desired or planned allocation $(x^*_{hi})_{h\in C,\, i\in N}$ as an equilibrium. We call this the *subsidies problem.* We first study the case without location externalities, where a vector of preferences $\boldsymbol{z} = (z_{hi})_{h\in C,\, i\in N}$ is exogenous, and then we extend the analysis for the general case in Section 3.2, where $\boldsymbol{z} = \boldsymbol{z}(\boldsymbol{x}^*)$.

3.1 *Analysis of the subsidy problem*

Inspired from Section 2, Theorem 3.1 provides a general result to calculate subsidies in order to reach any desired allocation $\boldsymbol{x}^* = (x^*_{hi})_{h\in C,\, i\in N}$ and vectors $\boldsymbol{\beta} = (\beta_h)_{h\in C}$ and $\boldsymbol{\rho} = (\rho_i)_{i\in N}$ as an equilibrium, obtained when atributes $\boldsymbol{z}$ are replaced by $\boldsymbol{z} + \boldsymbol{s}$. This result will then be useful for the case when $\boldsymbol{\beta}$ and $\boldsymbol{\rho}$ are the vectors of utilities and rents obtained at the equilibrium.

Theorem 3.1. *For any vector* $\boldsymbol{z} = (z_{hi})_{h\in C,\, i\in N}$*, let* $(b_h(\boldsymbol{z}))_{h\in C}$ *and* $(r_i(\boldsymbol{z}))_{i\in N}$ *be the unique solution to the dual market equilibrium problem* (2.8) *under the additional constraint* $b_1 = 0$*. Let* $(x_{hi}(\boldsymbol{z}))_{h\in C,\, i\in N}$*, given by* $x_{hi}(\boldsymbol{z}) = e^{\mu(z_{hi}-b_h(\boldsymbol{z})-r_i(\boldsymbol{z}))}$*, be the corresponding unique solution to the primal equilibrium problem* (2.7)*. Let* $(x^*_{hi})_{h\in C, i\in N} \in \Xi$ *be a desired allocation, let* $\boldsymbol{\beta} = (\beta_h)_{h\in C} \in \mathrm{R}^{|C|}$ *with* $\beta_1 = 0$ *and* $\boldsymbol{\rho} = (\rho_i)_{i\in N} \in \mathrm{R}^{|N|}$ *be arbitrary vectors. By defining*

$$(\forall h \in C)(\forall i \in N) \quad s_{hi}(\beta_h, \rho_i) = \frac{1}{\mu} \ln x^*_{hi} + \beta_h + \rho_i - z_{hi}, \tag{3.1}$$

and denoting by $\boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho}) = (s_{hi}(\beta_h, \rho_i))_{h\in C,\, i\in N}$*, we have, for every* $h \in C$ *and* $i \in N$*,* *that* $b_h(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))=\beta_h$, $r_i(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))=\rho_i$, *and* $x_{hi}(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))=x^*_{hi}$.

Proof. Let $\boldsymbol{\beta}=(\beta_h)_{h\in C}\in \mathrm{R}^{|C|}$ and $\boldsymbol{\rho}=(\rho_i)_{i\in N}\in \mathrm{R}^{|N|}$ be any vector satisfying $\beta_1=0$. It follows from Example 2.3 that $(b_h(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho})))_{h\in C}$ and $(r_i(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho})))_{i\in N}$ are the unique solution to the non-linear system

$$(\forall h\in C)(\forall i\in N)\quad \begin{cases}\sum_{i\in N} e^{\mu(z_{hi}+s_{hi}(\beta_h,\rho_i)-b_h(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))-r_i(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho})))}=H_h\\ \sum_{h\in C} e^{\mu(z_{hi}+s_{hi}(\beta_h,\rho_i)-b_h(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))-r_i(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho})))}=S_i.\end{cases} \tag{3.2}$$

It follows from (3.1) that (3.2) is equivalent to

$$(\forall h\in C)(\forall i\in N)\quad \begin{cases}\sum_{i\in N} x^*_{hi}e^{\mu(\beta_h-b_h(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))+\rho_i-r_i(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho})))}=H_h\\ \sum_{h\in C} x^*_{hi}e^{\mu(\beta_h-b_h(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))+\rho_i-r_i(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho})))}=S_i.\end{cases} \tag{3.3}$$

Hence, since $(x^*_{hi})_{h\in C,i\in N}\in\Xi$ we have that the unique solution to the system under the constraint $\beta_1=0$ is, for every $h\in C$ and $i\in N$, $b_h(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))=\beta_h$ and $r_i(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))=\rho_i$. Moreover, we deduce from (3.1) that, for every $h\in C$ and $i\in N$,

$$x_{hi}(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))=e^{\mu(z_{hi}+s_{hi}(\beta_h,\rho_i)-b_h(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))-r_i(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho})))}=e^{\mu(z_{hi}+s_{hi}(\beta_h,\rho_i)-\beta_h-\rho_i)}=x^*_{hi}, \tag{3.4}$$

which yields the result. This means that, by considering the modified vector of utilities $\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho})$ instead of $\boldsymbol{z}$, the land use equilibrium allocation coincides with the desired allocation $\boldsymbol{x}^*$ and the equilibrium utilities and rents coincide with the arbitrary chosen ones. □

Remark 3.2. *Note that vectors* $(\beta_h)_{h\in C}$ *and* $(\rho_i)_{i\in N}$ *are arbitrary in the statement of Theorem 3.1. However, the equilibrium conditions in equation* (3.4) *imply that they represent mutually dependent rents and utilities. Therefore, the subsidies computed in* (3.1) *allows the planner, in addition to reach the optimal allocation, the flexibility to impose any vectors of rents and utilities.*

Remark 3.3. *Theorem 3.1 asserts that optimal subsidies depends on* $\boldsymbol{x}^*$ *and* $\boldsymbol{z}$ *but not on* $\boldsymbol{x}(\boldsymbol{z})$ *solution to the primal equilibrium problem* (2.7)*. Nevertheless, the optimal allocation* $\boldsymbol{x}^*=\boldsymbol{x}(\boldsymbol{z}+\boldsymbol{s})$ *satisfy the equilibrium conditions* (2.9) *when optimal subsidies are considered.*

3.2 *The case with location externalities*

Location externalities are associated with the perception of agents regarding neighborhood quality, which depends on the location of other agents. Then, $\boldsymbol{z}=\boldsymbol{z}(\boldsymbol{x})$ and the equilibrium conditions $x_{hi}=e^{\mu(z_{hi}(\boldsymbol{x})-b_h(\boldsymbol{z}(\boldsymbol{x}))-r_i(\boldsymbol{z}(\boldsymbol{x})))}$ defines a fixed point and it is no longer solution to (2.7).

Given an objective allocation $\boldsymbol{x}^*$ and the endogeneity of location externalities, we aim at finding optimal subsidies $\boldsymbol{s}$ satisfying the condition $x^*_{hi}=e^{\mu(z_{hi}(\boldsymbol{x}^*)+s_{hi}-b_h(\boldsymbol{z}(\boldsymbol{x}^*)+\boldsymbol{s})-r_i(\boldsymbol{z}(\boldsymbol{x}^*)+\boldsymbol{s}))}$.

Note that Remark 3.3 states that optimal subsidies are dependent on $\boldsymbol{x}^*$ and $\boldsymbol{z}$ but are independent of the equilibrium $\boldsymbol{x}(\boldsymbol{z})$. We now show that this is also true in the case with externalities because $\boldsymbol{x}^*$ is exogenous in the computation of subsidies. Indeed, by replacing $\boldsymbol{z}$ by $\boldsymbol{z}(\boldsymbol{x}^*)$ in (3.1) we obtain

$$(\forall h \in C)(\forall i \in N) \quad s_{hi}(\beta_h, \rho_i) = \frac{1}{\mu} \ln x^*_{hi} + \beta_h + \rho_i - z_{hi}(\boldsymbol{x}^*) \tag{3.5}$$

and replacing this subsidies in (3.3) and (3.4), we deduce

$$(\forall h \in C) \quad b_h(\boldsymbol{z}(\boldsymbol{x}^*) + \boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho})) = \beta_h \tag{3.6}$$

$$(\forall i \in N) \quad r_i(\boldsymbol{z}(\boldsymbol{x}^*) + \boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho})) = \rho_i \tag{3.7}$$

$$(\forall (h,i) \in C \times N) \quad x_{hi}(\boldsymbol{z}(\boldsymbol{x}^*) + \boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho})) = x^*_{hi}. \tag{3.8}$$

From these equations, we deduce that subsidies depend only on the exogenous $\boldsymbol{x}^*$ and also that (3.8) is exactly the equilibrium condition in the case with externalities. Additionally, we note that the interpretation of (3.8) is that subsidies are chosen in order to gurantee this condition for an exogenous $\boldsymbol{x}^*$ and, therefore, this equation does not represent a fixed point in $\boldsymbol{x}^*$.

3.3 *Policy options for the computation of subsidies*

The application of Theorem 3.1 requires the planner to be able to identify vectors $(\beta_h)_{h\in C}$ and $(\rho_i)_{i\in N}$ in order to compute the subsidies (which includes taxes) for reaching the optimum $(x^*_{hi})_{h\in C,\, i\in N}$. Since these vectors can be chosen arbitrarily with $\beta_1 = 0$, there are infinite ways to define them, therefore we now propose some examples of plausible criteria to help in this task.

Example 3.4 (Maintain rents and utilities as without subsidies)**.** *In this case the policy is to maintain the rents and utilities associated to the market equilibrium without subsidies. For this purpose, the planner should set $\beta_h = b_h(\boldsymbol{z})$ and $\rho_i = r_i(\boldsymbol{z})$, for every $h \in C$ and $i \in N$, which are the utilities and rents obtained by solving the equilibrium problem* (2.8) *under the constraint $\beta_1 = 0$ without subsidies. Then, Theorem 3.1 asserts that the optimal subsidies to attain $(x^*_{hi})_{h\in C\, i\in N}$ are given in this case by*

$$(\forall h \in C)(\forall i \in N) \quad s_{hi} = \frac{1}{\mu} \ln x^*_{hi} + b_h(\boldsymbol{z}) + r_i(\boldsymbol{z}) - z_{hi}. \tag{3.9}$$

Example 3.5 (Minimize resource transferences under social disturbance constraints)**.** *The previous case may yield inacceptably high levels of subsidies/taxes. Then, we examine the option of relaxing the condition of maintaining the rents and utilities without subsidies, allowing to change around them in order to reduce the impact of the policy. We define the social disturbance as $SD(\boldsymbol{\beta}, \boldsymbol{\rho}) = \sum_{h\in C}(\beta_h - b_h(\boldsymbol{z}))^2 + \sum_{i\in N}(\rho_i - r_i(\boldsymbol{z}))^2$, which represents the deviation of utilities and rents with respect to those attained at the equilibrium. By considering a gap $\varepsilon > 0$, we propose the planner to*

$$\begin{aligned} &\min_{\boldsymbol{\beta},\boldsymbol{\rho}} \sum_{h\in C} \sum_{i\in N} s^2_{hi}(\beta_h, \rho_i) \\ &s.t.\ SD(\boldsymbol{\beta}, \boldsymbol{\rho}) \leq \varepsilon, \end{aligned} \tag{3.10}$$

which yields a vector $(\boldsymbol{\beta}^*, \boldsymbol{\rho}^*)$*, and to compute the subsidies*

$$(\forall h \in C)(\forall i \in N) \quad s_{hi} = \frac{1}{\mu} \ln x^*_{hi} + \beta^*_h + \rho^*_i - z_{hi}. \tag{3.11}$$

Example 3.6 (Minimize social disturbance under budgets constraints)**.** *Now the planner aims to minimize the social disturbance generated by the subsidies policies* $s(\boldsymbol{\beta}, \boldsymbol{\rho}) = (s_{hi}(\beta_h, \rho_i))_{h \in C,\, i \in N}$ *under individuals and system budget constraints. Let, for every* $h \in C$*,* $I_h \in [0, +\infty[$*, and let* $I \in]0, +\infty[$*. The problem is to*

$$\begin{aligned} &\underset{\boldsymbol{\beta},\boldsymbol{\rho}}{\text{minimize}} \;\; SD(\boldsymbol{\beta}, \boldsymbol{\rho}) \\ s.t. \quad & -\sum_{i \in N} s_{hi}(\beta_h, \rho_i) \leq I_h, \quad \forall h \in C \\ & \sum_{h \in C} \sum_{i \in N} s_{hi}(\beta_h, \rho_i) \leq I. \end{aligned} \tag{3.12}$$

This problem includes budgetary constraints for each type of household and for the planner, where, for every $h \in C$*,* I_h *is the representative income of households type* h*, and* I *respresents the maximum amount of money that the planner wish to use in the subsidies policies. Given a solution* $(\boldsymbol{\beta}^*, \boldsymbol{\rho}^*)$ *to* (3.12)*, subsidies are computed via* (3.11)*. Note that, since it is always possible to find vectors* $\boldsymbol{\beta}$ *and* $\boldsymbol{\rho}$ *such that* $s_{hi}(\beta_h, \rho_i) = 0$ *for every* $h \in C$ *and* $i \in I$*, problem* (3.12) *is always feasible.*

Example 3.7 (Constraints on subsidies)**.** *In this example we describe some policies allowing the policy maker to assume some external constraints on his flexibility to set subsidies. In each case we identify a choice of vectors* $(\beta_h)_{h \in C}$ *and* $(\rho_i)_{i \in N}$ *from which we compute subsidies from* (3.1) *satisfying the desired constraint. For this purpose, instead of* $\beta_1 = 0$*, we consider conditions (i)-(iv) in Proposition 2.4 for achieving uniqueness of the dual variables to allow imposing the specific subsidy constraints. Given* $\eta \in \mathrm{R}$*, the following policies are considered:*

(i) Agent type 1 is not affected*: For obtaining* $s_{1i}(\beta_1, \rho_i) = 0$*, for every* $i \in N$*, it is enough to choose* $(\beta_h)_{h \in C} \in D_1 = \left\{(\beta_h)_{h \in C} \in \mathrm{R}^{|C|} \;\middle|\; \beta_1 = \eta\right\}$*, and set*

$$(\forall i \in N) \quad \rho_i = z_{1i} - \frac{1}{\mu} \ln x^*_{1i} - \eta. \tag{3.13}$$

(ii) Zone 1 is not affected*: For obtaining* $s_{h1}(\beta_h, \rho_1) = 0$*, for every* $h \in C$*, it is enough to choose* $(\rho_i)_{i \in N} \in D_3 = \left\{(\rho_i)_{i \in N} \in \mathrm{R}^{|N|} \;\middle|\; \rho_1 = \eta\right\}$*, and set*

$$(\forall h \in C) \quad \beta_h = z_{h1} - \frac{1}{\mu} \ln x^*_{h1} - \eta. \tag{3.14}$$

(iii) Self funded policy by household type*: For obtaining* $\sum_{i \in N} s_{hi}(\beta_h, \rho_i) = 0$*, for every* $h \in C$*, it is enough to choose* $(\rho_i)_{i \in N} \in D_4 = \left\{(\rho_i)_{i \in N} \in \mathrm{R}^{|N|} \;\middle|\; \frac{1}{|N|} \sum_{i \in N} \rho_i = \eta\right\}$*, and set*

$$(\forall h \in C) \quad \beta_h = \frac{1}{|N|} \sum_{i \in N} \left(z_{hi} - \frac{1}{\mu} \ln x^*_{hi}\right) - \eta. \tag{3.15}$$

(iv) Self funded policy by zone*: For obtaining $\sum_{h\in C} s_{hi}(\beta_h,\rho_i)=0$, for every $i\in N$, it is enough to choose $(\beta_h)_{h\in C}\in D_2=\big\{(\beta_h)_{h\in C}\in \mathrm{R}^{|C|}\;\big|\;\frac{1}{|C|}\sum_{h\in C}\beta_h=\eta\big\}$, and set*

$$(\forall i\in N)\quad \rho_i=\frac{1}{|C|}\sum_{h\in C}\Big(z_{hi}-\frac{1}{\mu}\ln x^*_{hi}\Big)-\eta. \tag{3.16}$$

4. APPLICATION TO THE SOCIAL INCLUSION PROBLEM

In this section we present the application of land use planning problem and the associated optimal subsidies problem, to the classical example of a society concerned about social exclusion. We consider the case where the policy maker seeks the combined objective of maximizing agents utilities and minimizing a measure of spatial socioeconomic heterogeneity.

4.1 *Analysis of the problem*

Consider the definitions and notations introduced in Section 2 and the following measure for the socioeconomic homogeneity for an allocation $\boldsymbol{x}=(x_{hi})_{h\in C,\,i\in N}\in\Xi$

$$SE(\boldsymbol{x})=\sum_{i\in N}\left(\frac{\sum_{h\in C}x_{hi}I_h}{S_i}-\overline{I}\right)^2, \tag{4.1}$$

which measures the deviation of an average exclusion index in each zone for an allocation $\boldsymbol{x}$ with respect to the city average $\overline{I}=\sum_{h\in C}H_hI_h/T$. Here, for every $h\in C$, $I_h\in\,]0,+\infty[$ is the index of exclusion of households type h, e.g. this index may be household's income. Instead of using this measure, which is not convex, in the following proposition we provide a related measure which is separable and convex as the objective function in (2.4).

Proposition 4.1. *Let $\boldsymbol{x}=(x_{hi})_{h\in C,\,i\in N}\in\Xi$ and define the zone segregation level and the aggregated segregation level by*

$$(\forall i\in N)\quad SL_i(\boldsymbol{x})=\sum_{h\in C}I_h(x_{hi}/S_i-H_h/T)^2\quad\text{and}\quad SL(\boldsymbol{x})=\sum_{i\in N}SL_i(\boldsymbol{x}), \tag{4.2}$$

respectively. Then, $0\le SE(\boldsymbol{x})\le(\sum_{h\in C}I_h)SL(\boldsymbol{x})$ and the unique minimizer of SL, $\boldsymbol{x}_{SL}=\big((S_iH_h)/T\big)_{h\in C,\,i\in N}$, is a minimizer of SE.

Proof. Since the SL is strictly convex and coercive it has a unique minimizer $\boldsymbol{x}_{SL}\in\Xi$. Let $\boldsymbol{x}^*=\big((S_iH_h)/T\big)_{h\in C,\,i\in N}$. Since $SL(\boldsymbol{x}^*)=0$ and SL is a positive function, it is clear that $\boldsymbol{x}_{SL}=\boldsymbol{x}^*$. Moreover, let $\boldsymbol{x}=(x_{hi})_{h\in C,\,i\in N}\in\Xi$. It follows from (4.1), (4.2), and Bauschke and Combettes (2011, Lemma 2.13(ii)) that

$$0\le SI(\boldsymbol{x})=\sum_{i\in N}\left(\sum_{h\in C}I_h(x_{hi}/S_i-H_h/T)\right)^2\le\left(\sum_{h\in C}I_h\right)SL(\boldsymbol{x}) \tag{4.3}$$

and, hence, $SI(\boldsymbol{x}_{SL})=0$, which yields the result. □

Then, the problem under consideration in this section is to find an allocation which minimizes the aggregated segregation level and, simultaneously, maximizes the total utility. More precisely,

$$\underset{\boldsymbol{x}\in\Xi\cap\mathbb{R}_+^{|C|\times|N|}}{\text{minimize}} \quad -\sum_{h\in C}\sum_{i\in N} x_{hi}z_{hi} + \frac{1}{\alpha}\sum_{h\in C}\sum_{i\in N} I_h(x_{hi}/S_i - H_h/T)^2, \tag{4.4}$$

where Ξ is defined in (2.2) (market clearing) and $\alpha > 0$. This parameter is a measure of the relative importance of the utility of households compared with segregation objective, i.e. the higher is α, the higher is the importance of the utility compared with the segregation for the planner.

Problem (4.4) is a particular case of Problem 2.1 when, for every $(h,i) \in C \times N$, $\psi_{hi}(z_{hi},\cdot)\colon x \mapsto -xz_{hi} + I_h(x/S_i - H_h/T)^2/\alpha$. Note that functions $(\psi_{hi})_{h\in C, i\in N}$ are in $\mathscr{C}$. Therefore, it follows from Proposition 2.4 that (4.4) has a unique primal solution $\boldsymbol{x}^*_\alpha(\boldsymbol{z})$, which we call *inclusion optimum*, and a unique dual solution for the pair utility and rent $(\boldsymbol{\gamma}^*_\alpha(\boldsymbol{z}), \boldsymbol{\delta}^*_\alpha(\boldsymbol{z}))$ under the additional condition $\gamma_{1,\alpha} = 0$.

We expect from (4.4) that the segregation level at the optimum is increasing with α. The next proposition asserts that this relationship is indeed quadratic.

Proposition 4.2. *If $\alpha > 0$ is small enough, then the dual solution does not depend on α, i.e., $(\boldsymbol{\gamma}^*_\alpha(\boldsymbol{z}), \boldsymbol{\delta}^*_\alpha(\boldsymbol{z})) = (\boldsymbol{\gamma}^*(\boldsymbol{z}), \boldsymbol{\delta}^*(\boldsymbol{z}))$, and the segregation level defined in* (4.2) *at the inclusion optimum is*

$$SL(\boldsymbol{x}^*_\alpha(\boldsymbol{z})) = \frac{\alpha^2}{4}\sum_{h\in C}\sum_{i\in N}\frac{S_i^2}{I_h}(z_{hi} - \gamma^*_h(\boldsymbol{z}) - \delta^*_i(\boldsymbol{z})). \tag{4.5}$$

Proof. Suppose that $\boldsymbol{x}^*_\alpha(\boldsymbol{z})$ is strictly feasible, i.e., that, for every $h \in C$ and $i \in N$, $x^*_{hi,\alpha}(\boldsymbol{z}) > 0$. Then, the first order conditions of (4.4) yield

$$(\forall h \in C)(\forall i \in N) \quad x^*_{hi,\alpha} = \frac{H_h S_i}{T} + \alpha\frac{S_i^2}{2I_h}(z_{hi} - \gamma^*_{h,\alpha}(\boldsymbol{z}) - \delta^*_{i,\alpha}(\boldsymbol{z})), \tag{4.6}$$

where $(\gamma^*_{h,\alpha}(\boldsymbol{z}))_{h\in C}$ and $(\delta^*_{i,\alpha}(\boldsymbol{z}))_{i\in N}$ are the Lagrange multipliers of the constraints in Ξ (dual solution). Imposing these constraints on the primal solution $(x^*_{hi,\alpha})_{h\in C, i\in N}$ we obtain

$$(\forall i \in N) \quad S_i = \sum_{h\in C} x^*_{hi,\alpha} = S_i + \alpha\frac{S_i^2}{2}\sum_{h\in C} I_h^{-1}(z_{hi} - \gamma^*_{h,\alpha}(\boldsymbol{z}) - \delta^*_{i,\alpha}(\boldsymbol{z})) \tag{4.7}$$

$$(\forall h \in C) \quad H_h = \sum_{i\in N} x^*_{hi,\alpha} = H_h + \alpha\frac{1}{2I_h}\sum_{i\in N} S_i^2(z_{hi} - \gamma^*_{h,\alpha}(\boldsymbol{z}) - \delta^*_{i,\alpha}(\boldsymbol{z})), \tag{4.8}$$

which yields that, under the additional condition $\delta_1 = 0$, the dual solution is the unique solution to the system

$$(\forall h \in C) \quad \gamma_h = \sum_{i\in N}(z_{hi} - \delta_i)\eta_i \tag{4.9}$$

$$(\forall i \in N) \quad \delta_i = \sum_{h\in C}(z_{hi} - \gamma_h)\iota_h, \tag{4.10}$$

where, for every $h \in C$ and $i \in N$, $\iota_h = I_h^{-1}/\sum_{g\in C} I_g^{-1}$ and $\eta_i = S_i^2/\sum_{j\in N} S_j^2$. Hence, the dual solution $(\boldsymbol{\gamma}^*, \boldsymbol{\delta}^*)$ does not depend on α and, therefore, from (4.6), α should be small enough for satisfying $x^*_{hi,\alpha}(\boldsymbol{z}) > 0$. Finally, (4.5) follows from a straightforward computation. □

4.2 *Numerical example*

We provide an example in a fictitious city with high income segregation, where we obtain the inclusion optimum, we compare it with the equilibrium and we compute subsidies in order to reach this optimum. We show that the solution to the inclusion problem is an allocation with better levels of socioeconomic homogeneity. In this example, we take arbitrary values for the supply, demand (satisfying market clearing condition), and utility.

Consider a city with 10 zones ($|N| = 10$) and 5 types of households ($|C| = 5$). The convergence criteria of the algorithm for solving problem (2.7) is $\|\boldsymbol{b}_n - \boldsymbol{b}_{n+1}\|/\|\boldsymbol{b}_n\| \leq 10^{-10}$ and $\|\boldsymbol{r}_n - \boldsymbol{r}_{n+1}\|/\|\boldsymbol{r}_n\| \leq 10^{-10}$. The real estate supply per zone and the number of households per type are $S = (S_1, \ldots, S_{10}) = (25, 37, 24, 21, 34, 43, 23, 27, 20, 14)$ and $H = (H_1, \ldots, H_5) = (50, 56, 51, 60, 51)$, respectively, and the total supply (or demand) is $T = 268$. The average income of households per type is $I = (I_1, \ldots, I_5) = (2, 4, 6, 8, 10)$, which is used in this case as the segregation index in (4.1) and (4.2). Additionally, the utilities $\boldsymbol{z} = (z_{hi})_{h\in C,\, i\in N}$ are presented in Table 1. We recall that, for every $h \in C$ and $i \in N$, z_{hi} represents the utility perceived by a household type h for a location in i.

Table 1. Exogenous values for $(z_{hi})_{h\in C,\, i\in N}$

h \ i	1	2	3	4	5	6	7	8	9	10
1	50	50	50	0	0	0	0	-50	-50	-50
2	50	50	0	0	0	0	0	-50	-50	-50
3	-50	-50	0	0	50	50	50	50	0	0
4	0	0	0	50	50	50	0	0	0	0
5	-50	-50	-50	0	0	0	50	50	50	50

Table 2 presents the equilibrium $\overline{\boldsymbol{x}}(\boldsymbol{z}) = (\overline{x}_{hi}(\boldsymbol{z}))_{h\in C,\, i\in N}$ obtained by solving (2.7) via the algorithm in Macgill (1977) with $\mu = 5 \times 10^{-2}$. It also shows the income segregation level of the equilibrium by zone and in the whole city computed by (4.2). Additionally, in Figure 1 we show the proportion of households of each type $h \in \{1, \ldots, 5\}$ located in every zone $i \in \{1, \ldots, 10\}$ at the equilibrium. We observe very high segregation in all zones concentrating rich and poor households in different zones.

Table 2. Equilibrium $\overline{\boldsymbol{x}}(\boldsymbol{z})$ and segregation level.

h \ i	1	2	3	4	5	6	7	8	9	10	$SL(\overline{\boldsymbol{x}})$
1	9	13	19	2	2	3	2	0	0	0	
2	16	23	3	3	3	4	3	0	0	0	
3	0	0	1	1	11	14	9	13	1	1	
4	0	1	1	15	16	21	1	2	2	1	
5	0	0	0	1	1	1	9	12	16	11	
$SL_i(\overline{\boldsymbol{x}})$	1.64	1.64	1.57	2.17	1.00	1.00	0.89	1.61	4.13	4.13	**19.78**

In order to obtain a less segregated city, we consider the computation of the aggregated segregation level in terms of α obtained from (4.5) in Proposition 4.2 for small values of α. Observing that for these values, parameters $(\boldsymbol{\gamma}, \boldsymbol{\delta})$ are independent of α in (4.5), the aggregated segregation level is a quadratic and increasing function of α as shown in Figure 2. Note that an aggregated segregation level of 20 units of income (approximately that of the equilibrium $\overline{\boldsymbol{x}}(\boldsymbol{z})$) is obtained by using a value of $\alpha \approx 1.3 \times 10^{-4}$ (see Figure 2).

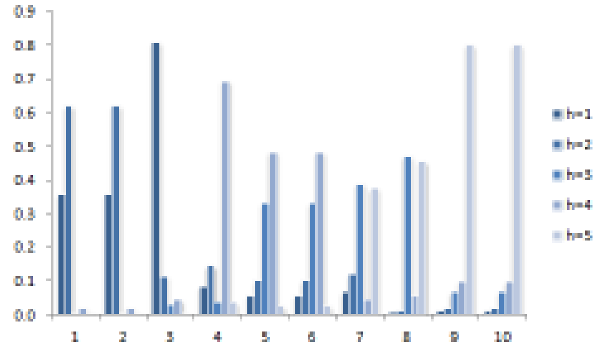

Figure 1. Proportion of types of households by zone for the equilibrium.

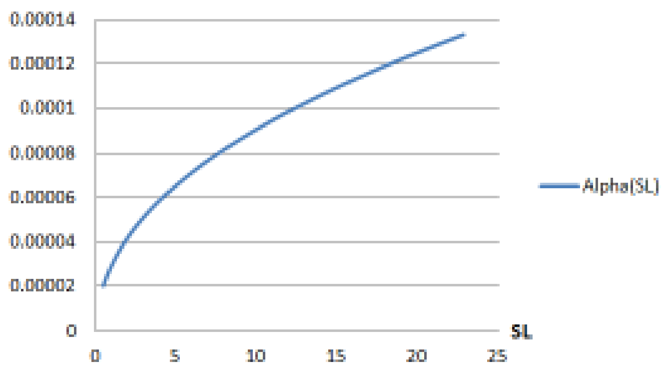

Figure 2. Function $\alpha(SL)$.

By considering the much lower value of $\alpha = 3 \times 10^{-5}$, we obtain the solution $\boldsymbol{x}^*(\boldsymbol{z}) = (x^*_{hi}(\boldsymbol{z}))_{h\in C,\, i\in N}$ to (4.4) from the algorithm (**??**). The result and the corresponding segregation level in every zone are presented in Table 3. In Figure 3 we exhibit the proportion of households of each type $h \in \{1, \dots, 5\}$ located in every zone $i \in \{1, \dots, 10\}$. We observe that the initial segregation level of the equilibrium $\overline{\boldsymbol{x}}(\boldsymbol{z})$ is drastically reduced in every zone and the total changes from 19.78 to 0.99. The reader can verify from Figure 2 that $\alpha(0.99) \approx 3 \times 10^{-5}$ which coincides with the theoretical square root relation for $\alpha(SL)$ provided in Proposition 4.2.

Table 3. Solution $\boldsymbol{x}^*$ and segregation level.

h \ i	1	2	3	4	5	6	7	8	9	10	$SL(\boldsymbol{x}^*)$
1	7	11	6	4	5	6	4	3	3	2	
2	7	13	5	4	6	7	4	4	3	2	
3	3	2	4	4	8	11	5	7	4	3	
4	5	7	5	5	9	12	4	5	4	3	
5	3	4	3	4	6	7	6	8	6	4	
$SL_i(\boldsymbol{x}^*)$	0.13	0.29	0.05	0.01	0.04	0.06	0.07	0.20	0.10	0.05	**0.99**

We then compute subsidies for reaching the inclusion optimum $\boldsymbol{x}^*(\boldsymbol{z})$. In Table 4 and 5 we present subsidies, rents, and households' utilities considering two policies mentionned in Section 3.3: Agent 1 is not affected and Self funded policy by zone. In the first policy, agents of type 1 are not affected by the policy, i.e., no subsidies or taxes, and we arbitrarily

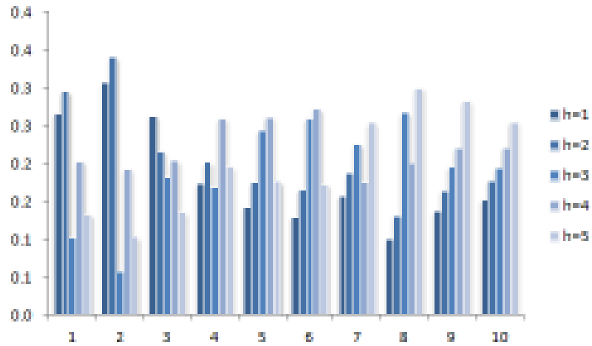

Figure 3. Proportion of types of households by zone for the inclusion optimum.

choose to set $\boldsymbol{\beta} = (0, 1, 1, 1, 1)$ and $\eta = 0$. The resulting rents and subsidies satisfying these conditions are shown in Table 4. Note that, this policy requires setting positive and negative subsidies, i.e., subsidies and taxes. Also note that the negative values on rents can be made positive by setting a negative value of $\eta \leq -64$, which is the minimum rent; of course, this reduces utility of households type 1. We illustrate the effects of chosing different η on the average of rents and utilites in Figure 4. The second policy allows the policy maker to achieve the inclusion optimum without investing in subsidies or collecting taxes in any zone, she only make transfers between agents in each zone. For this case we choose $\eta = 0$ and $\boldsymbol{\beta} = (0, 1, 1, 1, -3)$ satisfying $\frac{1}{|C|}\sum_{h \in C} \beta_h = \eta$ and the corresponding rents and subsidies are shown in Table 5. Again, note that negative rents can be made positive by chosing $\eta \leq -24$, which affects the average of utilities of households, without affecting the policy constraint. The effects of η in the average of rents and utilites is illustrated in Figure 5.

h \ i	1	2	3	4	5	6	7	8	9	10	β_h
1	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	0
2	2	2	51	2	2	2	2	2	2	2	1
3	98	97	50	1	-47	-47	-48	-96	-48	-48	1
4	51	51	51	-46	-46	-45	3	-46	-46	-47	1
5	99	98	99	1	2	2	-47	-95	-96	-97	1
ρ_i	33	29	34	-13	-18	-20	-14	-64	-62	-59	

Table 4. Subsidies for policy "Agent type 1 is not affected".

5. CONCLUSIONS

To the best of our knowledge, the land use planning and subsidies problems defined in a discrete domain of zones and households remains open, so planners have neither a method to identify the optimum allocation of households for specific objectives, nor how to calculate optimal subsidies.

The main contribution is to tackle the complexity of the interdependence among subsidies, location externalities and optimal allocation. Our approach splits the problem into planning and subsidies problems, where the planning problem is independent of subsidies, and the subsidies problem uses the fundamental result that any feasible allocation

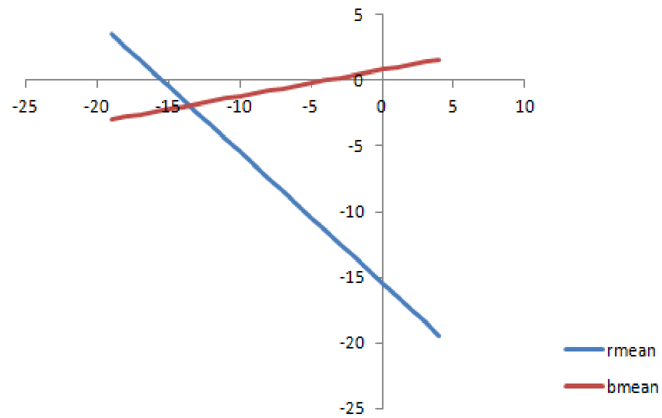

Figure 4. Average of rents and utilities as a function of η for policy "Agent type 1 is not affected".

$h \backslash i$	1	2	3	4	5	6	7	8	9	10	β_h
1	-49	-49	-49	9	19	18	19	48	38	39	0
2	-47	-47	2	11	21	21	21	50	41	41	1
3	49	48	0	10	-29	-28	-29	-48	-9	-9	1
4	2	2	2	-37	-27	-27	21	2	-8	-8	1
5	46	45	46	7	16	16	-32	-52	-62	-62	-3
ρ_i	-16	-20	-16	-4	1	-1	5	-17	-24	-20	
$\sum_{h \in C} s_{hi}^2$	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	

Table 5. Subsidies for policy "Self funded policy by zone".

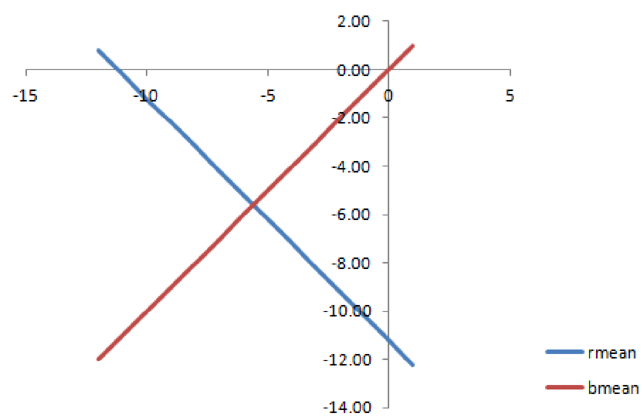

Figure 5. Average rents and utilities as a function of η for policy "Self funded policy by zone".

can be achieved by a specific set of subsidies (Theorem 3.1).

Additionally, we analyze a social inclusion problem as an interesting case in real cities, and we simulate a prototype city showing that methodology works. We also discuss different implementation policies that represent real situations where the policy makes faces constraints like budget or difficulty on subsidizing/taxing some zones and/or some households. This offers a flexibility usually useful in real cases.

The main limitations of our method is the assumption that travel decisions and transport costs are exogenous. On the other hand, a more realistic scenario will also include the behavior of suppliers and the excess of supply or demand. These extensions remain to be explored in further research.

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